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Stability Analysis for International Environmental Agreements with Social Externalities

BY

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Men may have insensibly acquired some gross ideas of mutual undertakings, and of the advantages of fulfilling them... If a deer was to be taken, every one saw that, in order to succeed, he must abide faithfully by his post: but if a hare happened to come within the reach of any one of them, it is not to be doubted that he pursued it without scruple, and, having seized his prey, cared very little, if by so doing he caused his companions to miss theirs.

Jean-Jaques Rousseau, *Discourse on the Origin and Foundations of Inequality among Men* (1755)

Abstract

The great part of environmental problems, like global warming, depletion of ozone layer or loss of biological diversity, are related to global commons and, for that, require global policies. During the last three decades, many times countries organized meetings to find an agreement on pollution control. The Montreal Protocol, the Kyoto Protocol or the several Conference of Parties (COP) in the last years are just few examples. At the same time, a big amount of scientist worked on the problem. Climatologist, ecologist and biologist tried and try to suggest possible solutions or design future scenarios.
But the problem is also an economic problem. Economists found a natural approach to the question in game theory. The first attempts set the issue in a static context (see e.g, [CS93], [Bar94]). But pollution is an evolving phenomenon, so a dynamic game approach can lead to more explicative results. Both in static and dynamic context, the literature is divided into two streams: cooperative and non-cooperative games. The main focus of the first stream is contrast the cooperative and non-cooperative solutions and show the benefits of cooperation. The real question in these games is how to allocate the payoff among players.
The non-cooperative stream starts from the consideration that there's no a supranational authority that can force countries to cooperate, so players choose non-cooperatively whether join or not in a coalition. In this games it is necessary to specify the concept of stability of the coalition. The terms most commonly used are those of internal and external stability (see [dAs+83]). In few words, these two conditions say that a coalition is stable if none of the members has an incentive to defect from cooperation and none of non-members has an incentive to join. So, the two focus points are research of the solutions (emissions or abatement level) and research of the coalition's dimension.
We consider myopic players, that is to say that economic interests are still too strong than environmental concern. It could be a limited point of view. There are relevant examples, like EU, that put into the foreground the control of emissions. But, with the arrival of the new millennium, the economic center of the world has changed his coordinates, and with it the center of environmental problems. The great challenge now is to include in emissions reduction process those countries that are not considered developed countries, but that give significant contribution to pollution (e.g, the countries called BRICS, Brazil, Russia, India, China, South Africa). We think that it's not realistic to ask to those countries to take care of environment for some kind of farsightedness or consider some kind of punishment for those who not cooperate.

So, we want try to design an IEA that's profitable. A classical result of the non-cooperative game theoretic literature supports only small coalition. But in reality, the principal agreements are signed by many countries.

The focus of this Thesis is to analyze if the presence of a Social Externality (see [CR06]), can lead to a large coalition. The hypothesis is that when countries have to make the decision to join or not an agreement, they consider all possible earnings due to relations with other countries.
We divide the world in two classes of countries, developed and developing, assuming that there is asymmetry between them, and homogeneity within each class.

In Chapter 1, we present some basic concepts of Dynamic Optimization and Game Theory. We first introduce the Optimal Control Theory, we show the Bolza problem and the Pontryagin Minimum Principle. Than, we extend the discussion to Dynamic Programming, comparing it and showing a method to find the optimal solutions.
Dynamic Programming is the method we use to find Nash equilibrium in dynamic context.
In the second part of the Chapter we introduce Game Theory, we give a definition of a Global Emission Game and Nash equilibrium, we introduce Dynamic Games and we discuss about Partial Cooperative Equilibrium. We conclude discussing about International Environmental Agreements, and some literature about them.

In Chapter 2, we present our first model. It is a static N-player game with asymmetric countries. We assume that a part of the players join in coalition, the rest acts non-cooperatively. We characterize emission solutions and than we discuss about stability assuming that the players are divided in two homogenous groups: developed and developing countries.
We also assume that a Social Externality affects the welfare of cooperators.

In Chapter 3, we introduce the Externality concept in a dynamic framework. We propose a Two-player differential game, in which player 1 represents developed countries, while player 2 represents developing countries. For developing country we assume a gradual involvement in environmental concern.
We find the feedback Nash equilibrium of the game and we discuss the stability of the agreement between the two players, adapting the definition of a self-enforcing agreements. Moreover, we do not impose that the cooperation starts from the outset of the game.

In Chapter 4, we extend the previous differential game assuming a world with N countries. We divide it in two asymmetric groups, developed and developing, maintaining the hypothesis of gradual involvement of developing countries. We characterize emission solutions both for cooperators and defectors, and we discuss about the size and composition of a stable coalition.

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Chapter 1

Preliminaries

In this Chapter we recall the Economic and Mathematical basis that we need to develop our models. So, we describe elements of Optimal Control Theory and Dynamic Programming, that are the frameworks and the instruments we use to find the Nash equilibrium in differential games.

Moreover we provide an introduction to Game Theory. We consider non-cooperative games, and we explain the notion of partial cooperative equilibrium. We discuss first about static games, introducing the basic concepts which define a game and the notion of Nash equilibrium, than we introduce dynamic game with continuos time.

Finally we introduce International Environmental Agreements (IEAs), with some literature and an explanation of the concept of Social Externality.

The Chapter is divided as follows: in Section 1 we introduce Dynamic Optimization; in Section 2 we discuss about Game Theory; in Section 3 we present International Environmental Agreements.

1.1 Dynamic Optimization

Dynamic Optimization is a mathematical discipline who deals with maximization or minimization problems in a dynamic context. One of the first examples in Calculus of Variations is the famous brachistocrone problem proposed by J. Bernoulli in 1696.
Then the theory has been greatly developed and applied in different disciplines, such as physics, engineering and economics. Calculus of Variations, Optimal Control Theory and Dynamic Programming are the principal areas in which we can divide Dynamic Optimization, and they are clearly linked between their. For our scopes, we move within the framework of Optimal Control Theory and Dynamic Programming.

For a brief introduction to Optimal Control Theory and Dynamic Programming see [Eva10]. For an exhaustive discussion see [KS91] and [FS06].

1.1.1 Introduction to Optimal Control Theory

Consider a differential equation that describes how a system evolves from a starting point. The state variable, also called trajectory, is the function we need to describe in which state is the system in every time t. Sometimes the state can be controlled by another variable, that we call control variable. So, mathematically, we have a controlled system:

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Assuming U to be a closed set of $\mathbb{R}^n$, we call a piecewise continuos function $u : [0, T] \to U \subset \mathbb{R}^n$ the *control variable* and $x(\cdot) \in A$, $A = \{x \in C^1_\star([0, T]; \mathbb{R}^n) : x(0) = x_0 \in \mathbb{R}^n\}$[1], the *state variable*, and $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$.
First of all, we assume standard hypothesis for the existence and the uniqueness of a solution of the system (1.1), as in the following modification of Picard Theorem:

Lemma 1.1. *Let $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ a Lipschitz continuous and sublinear in x function, uniformly with respect to t and u, that is there exists $K > 0$ such that*

[1] We denote with $C_\star([a, b]; \mathbb{R}^n)$ the space of piecewise continuos functions $f : [a, b] \to \mathbb{R}^n$ and with $C^1_\star([a, b]; \mathbb{R}^n)$ the space of piecewise continuosly differentiable functions $f : [a, b] \to \mathbb{R}^n$.

$\forall t \in [0,T]$, $u \in U$ and $x, y \in \mathbb{R}^n$

$$|f(t,x,u) - f(t,y,u)| \leq K|x-y|, \qquad |f(t,x,u)| \leq K(1+|x|). \tag{1.2}$$

Then for any fixed control $u \in C_\star([0,T],U)$ there exists a unique solution x to (1.1), that is a piecewise C^1 function that satisfies the system up to the discontinuity point of the control variable u.

We can interpret the solution of (1.1) as the dynamical evolution of the system. This function, that we should correctly write $x(\cdot, u(\cdot), x_0)$, depends upon the control $u(\cdot)$ and the initial point x_0. The issue now is: what is the optimal control u for the system (1.1)? To answer this question we need to introduce a "cost" criterion, that's a functional $J(u)$, in integral form:

$$J(u) = \int_0^T L(t, x(t), u(t))dt + \phi_T(x(T)).$$

So, we choose the control $u(\cdot)$ that is optimal in the sense that minimizes the functional $J(u)$.

Definition 1.2. Suppose that $U \subset \mathbb{R}^n$ is a closed set, $f \in C([0,T] \times \mathbb{R}^n \times U)$ satisfies the conditions (1.2) and that $L : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ are continuos and bounded from below functions. Then we call Bolza problem the following:

$$\inf_{u(\cdot)} J(u) = \int_0^T L(t, x^{x_0}(t), u(t))dt + \phi_T(x^{x_0}(T)), \tag{1.3}$$

where $u \in C_\star([0,T],U)$ and $x^{x_0}(\cdot, u) \in C^1_\star([0,T],\mathbb{R}^n)$ is a solution of

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(0) = x_0. \end{cases} \tag{1.4}$$

The principle we use to find the optimal control is due to the work of Pontryagin and in order to show it, we want first introduce the concepts of *costate* and *Hamiltonian.* A costate is a function $p(\cdot)$ that we can interpret as a sort of Lagrange multiplier. The basic idea of Pontryagin principle is that if $u^\star(\cdot)$ is an optimal control then there exists a costate $p^\star(\cdot)$ that's optimal in some way. The Hamiltonian in control theory is a function, H, defined as follows:

$$H(x,p,u) := f(x,u)\cdot p + L(x,u).$$

So, for the Bolza problem we can formulate the Pontryagin Minimum Principle as follows:

Theorem 1.3 (Pontryagin Minimum Principle)**.** *Assume $u^\star(\cdot)$ is an optimal control and let $x^\star(\cdot)$ the corresponding trajectory. Then, there exists a function $p^\star : [0,T] \to \mathbb{R}^n$ such that*

$$\dot{x}^\star(t) = \nabla_p H(x^\star(t), p^\star(t), u^\star(t)), \tag{1.5}$$

$$\dot{p}^\star(t) = -\nabla_x H(x^\star(t), p^\star(t), u^\star(t)), \tag{1.6}$$

and

$$H(x^\star(t), p^\star(t), u^\star(t)) = \min_u H(x^\star(t), p^\star(t), u), \tag{1.7}$$

in any continuity point t of $u^\star$.
Moreover, we have the final condition:

$$p^\star(T) = \nabla\phi(x^\star(T)),$$

and that the mapping $t \mapsto H(x^\star(t), p^\star(t), u^\star(t))$ is constant.

Let us present some classical examples.

Example 1.1. *Consider the classical Nordhaus Model. So, we have a country near to election day. To maintain the power, the party in charge of the government takes decisions only on two economic variables: u, the unemployment rate, and π, the inflation rate. These two variables enter in a vote function $V = V(u,\pi)$, which represents how many votes the management of unemployment and inflation can bring to the ruling party. For the function V we assume that is a decreasing function of u and π:*

$$\frac{\partial V}{\partial u} < 0, \qquad \frac{\partial V}{\partial \pi} < 0.$$

Moreover, u and π are related through the Phillips tradeoff relation:

$$p(t) = \phi(u(t)) + a\pi(t),$$

where ϕ is a decreasing function, $a \in (0,1]$ is a constant and π is the expected rate of inflation, solution of the following differential equation:

$$\dot{\pi}(t) = b(p(t) - \pi(t)), \qquad b \textit{ is a positive constant.}$$

Assuming T to be the time left to election, the problem that the incumbent party has to resolve is:

$$\max_u \int_0^T V(u(t), \phi(u(t)))e^{rt}dt,$$

subject to

$$\begin{cases} \dot{\pi}(t) = b\phi(u(t)) + b(a-1)\pi(t) \\ \pi(0) = \pi_0, \end{cases}$$

where $r > 0$ and e^{rt} is a decay memory factor, that's to say that actions near to the vote are more important for the voters. In the Nordhaus framework

$$\phi(u) = j - ku, \qquad V(u,\pi) = -(u^2 + hp), \quad j, k, h > 0.$$

Substituting into the maximization problem and taking account of the sign of V, we can rewrite

$$\min_u \int_0^T [u^2(t) + h(j - ku(t) + a\pi(t))]e^{rt}dt,$$

subject to

$$\begin{cases} \dot{\pi}(t) = b[j - ku(t) + (a-1)\pi(t)] \\ \pi(0) = \pi_0. \end{cases}$$

The Hamiltonian of this problem is:

$$H(t, \pi, u, \lambda) = e^{rt}[u^2 + h(j - ku + a\pi)] + \lambda b[j - ku + (a-1)\pi].$$

So, if $u^\star$ is optimal, then we have

$$\frac{\partial H}{\partial u} = 2ue^{rt} - hke^{rt} - bk\lambda = 0.$$

Since $\partial^2 H/\partial u^2 = 2e^{rt} > 0$, then the unique minimum $u^\star$ is

$$u^\star(t) = \frac{hk}{2} + \frac{bk}{2}\lambda^\star(t)e^{-rt},$$

where the co-state $\lambda^\star(t)$ solves the differential equation

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial H}{\partial \pi}(t, \pi^\star(t), \lambda(t)) = ahe^{rt} + b(a-1)\lambda(t) \\ \lambda(T) = 0. \end{cases}$$

The solution of the above Cauchy problem is given by

$$\lambda^\star(t) = \frac{ah}{r + b(a-1)}[e^{(r+b(a-1))T+(1-a)bt} - e^{rt}].$$

Finally, we can find the exact expression of optimal unemployment rate substituting the above value of $\lambda^\star$ in $u^\star$

$$u^\star(t) = \frac{hk}{2(r + b(a-1))}\left(r - b + abe^{(r+b(a-1))(T-t)}\right),$$

that is a decreasing function of t, $\frac{\partial u^\star}{\partial t} < 0$ for every t, and that assume value $u^\star(T) = hk/2 > 0$. So, the optimal unemployment rate is always positive.

The second example is the problem of a firm that wants to expand his capacity.

Example 1.2. *We are in the Eisner-Strotz model, who analyses the problem of a firm that wants expand the machinery used for production processes in a fixed period $[0, T]$, where $T > 0$. Assuming that the plant size is directly correlated to the capital stock K, we call $\pi = \pi(K)$ the profit rate function associated with each plant size. To realize the expansion, the firm has to face a cost C, that's a function of the velocity of the increment of capital $\dot{K}$. The capital stock K satisfies the differential equation $\dot{K}(t) = I(t)$, where $I(t)$ represent the net investment and it is the control variable. Thus, the optimal control problem can be written*

$$\max_I \int_0^T [\pi(K(t)) - C(I(t))]e^{-\rho t} dt,$$

where K is solution of the controlled system

$$\begin{cases} \dot{K}(t) = I(t) \\ K(0) = K_0. \end{cases}$$

We assume that $\rho, K_0 > 0$ and $I : [0,T] \to \mathbb{R}$. In order to find the solution of the problem we need to give an expression for $\pi(K)$ and $C(I)$, thus we assume

$$\pi(K) = \alpha K - \beta K^2 \quad \textit{and} \quad C(I) = aI^2 + bI,$$

where α, β, a, b are given positive costants. Then, the pre-Hamiltonian for this problem is given by

$$H(t, K, I, \lambda) = \alpha K - \beta K^2 - aI^2 - bI + \lambda I.$$

The research of maximum value of H, respect to the control variable I, leads us to

$$I^\star(t) = \frac{\lambda^\star - b}{2a},$$

where the costate $\lambda^\star$ is a solution of the Hamiltonian system

$$\begin{cases} \dot{\lambda}(t) = -\alpha + 2\beta K(t) + \rho\lambda(t), \quad \lambda(T) = 0 \\ \dot{K}(t) = \frac{\lambda^\star - b}{2a}, \quad K(0) = K_0. \end{cases}$$

Solving the Hamiltonian system, we can easily find the optimal path for capital stock $K^\star$:

$$K^\star(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \bar{K},$$

where

$$\begin{cases} \bar{K} = \frac{\alpha - \rho b}{2\beta}, \\ r_1 = \frac{\rho a + \sqrt{\rho^2 a^2 + 4a\beta}}{2a}, \qquad r_2 = \frac{\rho a - \sqrt{\rho^2 a^2 + 4a\beta}}{2a} \\ c_1 = \frac{(K_0 - \bar{K}) r_2 e^{r_2 T} + b/2a}{r_2 e^{r_2 T} - r_1 e^{r_1 T}}, \qquad c_2 = -\frac{(K_0 - \bar{K}) r_1 e^{r_1 T} + b/2a}{r_2 e^{r_2 T} - r_1 e^{r_1 T}}. \end{cases}$$

1.1.2 Dynamic Programming

Sometimes, in Mathematics, try to solve directly a problem could be not the easier way to face it. Embedding it in a larger class of problems and solve this one, could be a better method to work. Basically, this is what Dynamic Programming does for optimal control problems. So, we now consider the controlled dynamics

$$\begin{cases} \dot{\gamma}(s) = f(s, \gamma(s), u(s)) \qquad (0 < s < T) \\ \gamma(0) = x_0. \end{cases}$$

in which $\gamma(\cdot)$ is the state variable, with $\gamma \in [C^1_\star([0,T];\mathbb{R}^n) : \gamma(0) = x_0 \in R^n]$, $u : [0,T] \to U \subset \mathbb{R}^n$ is the control variable, with U let a closed subset of $\mathbb{R}^n$ and $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, which respects the conditions (1.2).
We consider also the associated payoff functional J

$$J(u) = \int_0^T L(\gamma(s), u(s))ds + \phi(\gamma(T))$$

in which $L : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ are continuos and bounded from below functions. What we do in Dynamic Programming is to consider a larger class of problems, obtained by letting vary the starting times and the starting points:

$$\begin{cases} \dot{\gamma}(s) = f(s, \gamma(s), u(s)) \qquad (t < s < T) \\ \gamma(t) = x, \end{cases} \tag{1.8}$$

with

$$J_{t,x}(u(\cdot)) = \int_t^T L(\gamma(s), u(s))ds + \phi(\gamma(T)), \tag{1.9}$$

where $U \subset \mathbb{R}^m$ is a closed set, $u(\cdot) \in C_\star([s,T], U)$, L and ϕ are continuos and bounded from below functions and f satisfying (1.2). The Pontryagin Minimum Principle gives us a necessary condition for an optimal solution. In the case of unique solution, we are able to determine the optimal trajectory and the optimal control. Moreover, we find open loop controls, that is for every initial state we find $u = u(t)$.
With Dynamic Programming approach we are able to obtain a necessary and sufficient condition for optimality. The difference is that now we get controls $u = u(t,x)$, called closed loop (or feedback) controls. A closed loop is a control which depends not only on the initial state of the system, but on the state of the system in the whole time interval. In that sense, we can say that the controls receive a feedback from the system. So, we have that $u = u(t,x)$ solve the closed loop equation

$$\begin{cases} \dot{\gamma}(t) = f(t, \gamma(t), u(t, \gamma(t))) \\ \gamma(s) = x \end{cases}$$

and the related minimization problem (1.9) for any initial state. The control function $u = u(t, x)$ is called *closed loop control* or *feedback control*.
So, we need a method to solve the problem (1.9) for all the choices of starting times $0 \le t \le T$ and all the initial points $x \in \mathbb{R}^n$. First we have to define the value function associated to the problem.

Definition 1.4. Let $x \in \mathbb{R}^n$ and $0 \le t \le T$, $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is the function defined as:

$$v(t, x) = \inf_{u(\cdot)\in U} J_{t,x}(u(\cdot)) \qquad (x \in \mathbb{R}^n, 0 \le t \le T), \tag{1.10}$$

it is called the *value function* and represents the lowest payoff value if we start at $x \in \mathbb{R}^n$ at time t. Note that $v(T, x) = \phi(x)$.

The method of Dynamic Programming is due to the work of Richard E. Bellman, in the middle of twentieth century and the theorem for optimality bears his name.

Theorem 1.5 (Bellman's Optimality Principle). *Let $L : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ continuos and bounded from below functions, and $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$, which respects the conditions (1.2), than the following statements hold true*

1. *For any fixed $(s, x) \in [0, T] \times \mathbb{R}^n$ and any $u \in C_\star([0, T], U)$*

$$v(s, x) \le \int_s^r L(t, \gamma_u^\star(t), u(t))dt + v(r, \gamma_u^\star(r)) \qquad \forall r \in [s, T]$$

2. *$u^\star(\cdot)$ is optimal if and only if*

$$v(s, x) = \int_s^r L(t, \gamma_{u^\star}^\star(t), u^\star(t))dt + v(r, \gamma_{u^\star}^\star(r)) \qquad \forall r \in [s, T]. \tag{1.11}$$

Now, we want to connect the Bellman Principle to the Hamilton equation. So from the theorem 1.5, we have

$$v(t, \gamma(t)) = \inf_u \left[\int_t^s L(\xi, \gamma(\xi), u(\xi))d\xi + v(s, \gamma(s)) \right].$$

If we take $s = t + h$ we can rewrite the last equation as follows:

$$v(t, \gamma(t)) = \inf_u \left[\int_t^{t+h} L(\xi, \gamma(\xi), u(\xi))d\xi + v(t + h, \gamma(t + h)) \right].$$

We assume that the function v is differentiable in (t,x), then

$$\begin{aligned}&v(t+h,\gamma(t+h)) =\\&= v(t,\underbrace{\gamma(t)}_{=x}+v_t(t,\gamma(t))h+\nabla_v(t,\gamma(t))(\gamma(t+h)-\gamma(t))+o(h)+\underbrace{o(\gamma(t+h)-\gamma(t)).}_{\le c|h|}\end{aligned}$$

So, we can write

$$v(t,x)=\inf_u\left[\int_t^{t+h}L(\xi,\gamma(\xi),u(\xi))d\xi+v(t,x)+v_t(t,x)h+\nabla_v(t,x)(\gamma(t+h)-\gamma(t))+o(h)\right].$$

We can take $v(t,x)$ outside the inf, so what we have is

$$0=\inf_u\left[\int_t^{t+h}L(\xi,\gamma(\xi),u(\xi))d\xi+v_t(t,x)h+\nabla_v(t,x)(\gamma(t+h)-\gamma(t))+o(h)\right].$$

Now, we divide all the members of the equation for h:

$$0=\inf_u\left[\frac{1}{h}\int_t^{t+h}L(\xi,\gamma(\xi),u(\xi))d\xi+v_t(t,x)+\nabla_v(t,x)\frac{(\gamma(t+h)-\gamma(t))}{h}+o(h)\right],$$

from which we have that, when $h\to 0$:

- $\frac{\gamma(t+h)-\gamma(t)}{h} \quad \xrightarrow{h\to 0} \quad \dot\gamma(t)=f(t,\gamma(t),u(t))$,
- $\frac{1}{h}\int_t^{t+h}L(\xi,\gamma(\xi),u(\xi))d\xi \quad \xrightarrow{h\to 0} \quad L(t,\gamma(t),u(t))$.

Formally, when $h\to 0$, then $v(t,x)$ satisfies

$$v_t(t,x)+\inf_u[L(t,x,u)+\nabla_v(t,x)\cdot f(t,x,u)]=0.$$

If we define the Hamiltonian H, as follows:

$$H(t,x,q):=\inf_{u\in U}[q\cdot f(t,x,u)+L(t,x,u)],$$

then we can say that $v(t,x)$ solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases}v_t(t,x)+H(t,x,\nabla_v(t,x))=0\\ v(T,x)=\phi(\gamma(T))=\phi(x), \qquad t\in[0,T], \quad x\in\mathbb{R}^n,\end{cases} \tag{1.12}$$

where

$$v_t(t,x) = \frac{\partial v(t,x)}{\partial t}, \qquad \nabla_v(t,x) = \left(\frac{\partial v(t,x)}{\partial x_i}\right)_{1 \le i \le h}.$$

The method of Dynamic Programming works in two steps to determine an optimal feedback control $u^\star(\cdot)$.

Step one is related to solve the HJB equation and find the value function v.

Step two consists in the use of the value function v and the HJB partial differential equation to design an optimal control $u^\star(\cdot)$, as follows. First of all define for each point $x \in \mathbb{R}^n$ and for each time $0 \le t \le T$ a parameter value

$$u(t,x) = a \in U,$$

where the minimum in HJB is attained. In other word, we select $u(t,x)$ such that

$$v_t(t,x) + L(t,x,u(t,x)) + \nabla_v(t,x) \cdot f(t,x,u(t,x)) = 0.$$

After that, we solve the controlled system

$$\begin{cases} \dot{\gamma}^\star(s) = f(s,\gamma^\star(s),u(s,\gamma^\star(s))) & (t \le s \le T) \\ \gamma(t) = x. \end{cases}$$

Finally, we can define the feedback control

$$u^\star(s) := u(s,\gamma^\star(s)). \tag{1.13}$$

To conclude, we show that the feedback control defined in (1.13) is optimal, in the sense that

$$J_{t,x}(u^\star(\cdot)) = \inf_u J_{t,x}(u(\cdot)).$$

By the definition of the functional J, we have that

$$J_{t,x}(u^\star(\cdot)) = \int_t^T L(s,\gamma^\star(s),u^\star(s))ds + \phi(\gamma^\star(T)).$$

To prove the optimality of $u^\star(\cdot)$, we just need of the expressions of HJB equation (1.12) and of the feedback control (1.13).

So, we can write

$$\begin{aligned}
J_{t,x}(u^\star(\cdot)) =& \int_t^T (-v_t(s,\gamma^\star(s)) - f(\gamma^\star(s),u^\star(s)) \cdot \nabla_x v(s,\gamma^\star(s)))ds + \phi(\gamma^\star(T)) \\
=& - \int_t^T (v_t(s,\gamma^\star(s)) + \nabla_x v(s,\gamma^\star(s)) \cdot \dot{\gamma}^\star(s))ds + \phi(\gamma^\star(T)) \\
=& - \int_t^T \frac{\partial}{\partial s} v(s,\gamma^\star(s))ds + \phi(\gamma^\star(T)) \\
=& - v(T,\gamma^\star(T)) + v(t,\gamma^\star(t)) + \phi(\gamma^\star(T)) \\
=& - \phi(\gamma^\star(T)) + v(t,\gamma^\star(t)) + \phi(\gamma^\star(T)) \\
=& v(t,\gamma) = \inf_{u(\cdot)} J_{t,x}(u(\cdot)).
\end{aligned}$$

That's to say

$$J_{t,x}(u^\star(\cdot)) = \inf_{u(\cdot)} J_{t,x}(u(\cdot)),$$

and then $u^\star(\cdot)$ is an optimal control.

1.2 Game Theory: An Introduction

In this section we provide an introduction to basic concepts of Game Theory.
In neo-classical theory the fundamental theorems of welfare Economics gives, in the context of walrasian paradigm a formalization of Adam Smith's *invisible hand*. That's to say that if some institutional conditions are verified, than individuals that follow their own interest tend to an efficient allocation of resources. Nevertheless, in almost all social interactions the axioms of fundamental theorems are not verified, because the social payoff depends on the structure of social relationships: beliefs, preferences of individuals, the laws who transform actions in payoffs, etc.
The problem is that the walrasian paradigm provides a minimal representation of institutions, so it's not a useful method to analyze the coordination problems, that are classical issues in Economics. **Garret Hardin** [Har68], in a famous article make an examples about a group of sheperds who exploit a common until it become ruined. To describe this situation he coined the phrase *tragedy of the commons*, which become a classic metaphor. Hardin said that the tragedy of the commons is the refusal of the invisible hand. So, if an invisible hand leads the social interaction, then individual choices and social optimum are both achieved. But in situation like the tragedy of the commons, the private interest leads to bad consequences for individuals and public goods. The tragedy of the commons goes beyond bucolic, and it's applicable to a large class of economic problems. For this reason, since the middle of the last century, Game Theory became a powerful instrument to model these situations (a really good introduction in [Bow06]).
Game Theory is a branch of Applied Mathematics who develop models of *strategical interactions*. With strategical interactions we mean all the situations in which the actions who people (that we call players) make are dependent on the actions of other people, and everyone knows it.

We can make some classical distinction in Game Theory. The first is between *cooperative* and *non-cooperative games*. Cooperative game theory studies negotiations among rational agents who can make binding agreements about how to play the game. Now the emphasis is on the groups or coalitions of the players. The scope is to establish which coalition will form and how the agents will share the benefit that the coalition has, according to some ideas of fairness given by a set of desirable properties for the solution (axioms). In non-cooperative games the emphasis is mainly on the individual behavior: agents cannot commit themselves and perceive self-interest looking for their actions in order to achieve the most likely outcome of the game according to the rules of the game. Sometimes pre-play communication between agents is allowed, but in a non-cooperative game they are not able to make agreements except for those which are established by the rules of the game. Of course, the division in cooperative and non-cooperative games may not be so clear, and we can have games who present elements of both kinds (like really are our models, examples of partial cooperative games are in [MT08], [MT09], [CGL11]). For a complete discussion about Game Theory see [FT91], [Mou86] and [Owe95].

In the following we show the main properties of non-cooperative games.
The first element which characterize a non-cooperative games are *players*. With this term we describe the agents, that could be individual or a set of individuals, who compete in the game.
The second element are *strategies*, which are all the choices available to a player. In a given situation, any player knows a set of actions from which he must choose a single element. The decisions made by the players may depend on the information available to each player. If the players act only once and indipendentely of each other, than we are considering *static games*. If at least one of the player is allowed to use a strategy that depends on previous actions, than we are considering *dynamic games*.
The third element are payoff functions, that are real valued functions defined on the cartesian product of the strategy spaces measuring desirability of the possible outcomes of the game, e.g. the amounts of money the players may win or loose. Any decision maker chooses the best action according to his preferences, represented by his payoff function, among all the actions available to him (theory of *rational choice*). The players cannot communicate before acting and usually we assume that all the players know the structure of the game, and know that their opponents know it, and know that their opponents know that they know and so on (the structure of the game is *common knowledge*). Than, we can give a definition of a non-cooperative game, denoted by G, in normal form as follows

Definition 1.6 (non-cooperative game)**.** A game G is a triple:

$$G = (I, S_i, w_i),$$

where $I = \{1, \ldots, n\}$ is the set of players, S_i is the strategy space and w_i is the payoff function of player i.

We call strategy profile a vector $s = (s_1, s_2, \ldots, s_n)$, and we denote with $w_i(s)$ the payoff for a player i associated with the strategy profile s.
Than, the combined strategy space S is the cartesian product of all the strategy set S_i: $S = \times_i S_i$; and the combined payoff function $w : S \to \mathbb{R}^n$ is given by: $w(s) = (w_1(s), w_2(s), \ldots, w_n(s))$. The extensive form representation gives a more complete information of the game and of its rules than the normal one (e.g., the sequence of moves), thus should be good norm use it when possible. However, for our models there is not a significant difference of information between the two representations, so we use the normal form for its parsimony. Two classical examples in Game Theory are the prisoner dilemma and the battle of sexes.

Example 1.3 (Prisoner dilemma)**.** *Two suspects are arrested by the police. The police have insufficient evidence for a conviction and, having separated both prisoners, visit each of them to offer the same deal: if one testifies ("defects") for the prosecution against the other and the other remains silent, the betrayer goes free and the silent accomplice receives the full 30 year sentence. If both remain silent, both prisoners are sentenced to only 2 year in jail for a minor charge. If each betrays the other, each receives a 8 year sentence. Each prisoner must make the choice of whether to betray the other or to remain silent. The situation can be modeled as a two-person finite game* $G = (I, S_i, w_i)$*, where* $I = \{1, 2\}$*, the strategy set is* $S_i = \{NC, C\}$*, common to both players, where the choice* C *means confess and the payoffs represent the years sentence as in table 1.1. Each player wants to minimize his own payoff.*

TABLE 1.1: Prisoner dilemma game matrix

	C	NC
C	8,8	0,30
NC	30,0	2, 2

Example 1.4 (Battle of the sexes)**.** *A husband and wife wish to go out together rather than separately and they select among the opera (O) and the stadium (S). While the wife (player* 1*) prefers O, the husband (player* 2*) prefers S. The payoff represent how much they prefer the choice. Players in this game are payoff maximizing. The situation can*

be modeled as a two-person finite game $G = (I, S_i, w_i)$, where $I = \{1, 2\}$, the strategy set is $S_i = \{O, S\}$, common to both players, and the payoffs are given by

Table 1.2: Battle of sexes game matrix

	O	S
O	2,1	0,0
S	0,0	1, 2

In order to give a concept for equilibrium, we need to introduce the concept of best replies. Denote with s_{-i} a strategy profile that includes the strategies of all players less than the player i: $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. So, a best reply for a player i, to a strategy profile $s \in S$, is a strategy $s_i \in S_i$ such that no other strategy that player i can choose, gives him an higher payoff against s. The set of best replies to a strategy profile $s \in S$, is a correspondence $\beta_i : S \to S_i$ which maps each strategy profile $s \in S$ to a nonempty set

$$\beta_i(s) = \{h \in S_i : w_i(s_i, s_{-i}) \geq w_i(h_i, s_{-i}) \quad \forall h_i \in S_i\}.$$

We can define the combined best reply correspondences, $\beta : S \to S$, as follows

$$\beta(s) = \times_{i \in I} \beta_i(s).$$

We are now able to introduce a fundamental concept in Game Theory, the notion of Nash equilibrium ([Nas50b], [Nas50a] and [Nas51]).

Definition 1.7 (Nash equilibrium)**.** $s \in S$ is a Nash equilibrium if $s \in \beta(s)$.

Thus, in terms of best replies, a strategy profile $s \in S$ is a Nash equilibrium if it is a best reply to itself, that's to say if s is a fixed point of the strategy best reply correspondende β.

Not all games has a Nash equilibrium. So, we want introduce a theorem which give some conditions for the existence of Nash equilibrium in games with infinite strategy space.

Theorem 1.8 (Existence of Nash equilibrium)**.** *Consider a game in normal form, where a player i has a strategy space S_i that is a nonempty, compact and convex subset of an Euclidean space. If the payoff functions w_i are continuos in strategy profiles s and quasi concave in a pure strategies s_i, than there exists a pure-strategy Nash equilibrium.*

For a proof of the previous theorem see [FT91].

Let us introduce now a concept well known in International Environmental Agreements literature: the partial cooperative equilibrium.

Let $G = (I, S_i, w_i)$ be an n-player normal form game, where $I = \{1, 2, ..., n\}$ $(n \in \mathcal{N})$ is the players set. For each player $i \in I$ we have the strategy space S_i and the payoff function $w_i : S \to \mathbb{R}$, being $S = \times_i S_i$. We denote by $s_{i,j}$ the vector $(s_i, s_{i+1}, \dots, s_{j-1}, s_j)$ and by $S_{i,j}$ the cartesian product $S_i \times S_{i+1} \times \cdots \times S_{j-1} \times S_j$ for any $i, j \in \{1, ..., n\}$ and $i < j$. Here we do not precise any assumption on the strategy sets $S_1, \dots, S_n$ as well as on the payoff functions $w_1, \dots, w_n$. We assume the existence of a partial cooperative equilibrium (for existence results see [MT09] and [CGL11]). Suppose that a set C of k players participate in an agreement, $C = \{1, \dots, k\} \subset I$, while the other $(n-k)$ players act non-cooperatively. The number k is called the level of cooperation and we assume that is given.

Cooperators make their choices maximizing the aggregate welfare of the coalition members, i.e.

$$W_k(s_1, \dots, s_n) = \sum_{j=1}^{k} w_j(s_1, \dots, s_n). \tag{1.14}$$

The non-signatories play as singletons and choose their strategies as a Nash equilibrium with payoffs $w_{k+1}, \dots, w_n$. In the case $k = 0$ we lie in a non-cooperative game, while if $k = n$ we are in the case of full cooperation.

We make the Nash-Cournot assumption, so all players choose their strategy simultaneously, taking into account the optimality of the other players as in the Nash equilibrium concept. Given the level of cooperation k, the signatories choose their strategy $(y_1, y_2, \dots, y_k) = y_{1,k} \in S_{1,k}$ and the following $(n-k)$ players with payoffs w_i, $i = k+1, \dots, n$, do not participate to the agreement and play as singletons, all the players deciding together. More precisely, we look for a vector $s^{NC}(k) = (\bar{s}_{1,k}, \bar{s}_{k+1}, \dots, \bar{s}_n) \in S$ such that for any $i = k+1, \dots, n$

$$w_i(\bar{s}_1, \dots, \bar{s}_n) = \max_{y \in S_i} w_i(\bar{s}_{1,k}, \bar{s}_{k+1}, \bar{s}_{i-1}, y, \bar{s}_{i+1}, \dots, \bar{s}_n)$$

and also

$$W_k(\bar{x}_1, \dots, \bar{x}_n) = \max_{y_{1,k} \in S_{1,k}} W_k(y_{1,k}, \bar{s}_{k+1}, \dots, \bar{s}_n) =$$

$$\max_{y_{1,k} \in S_{1,k}} \sum_{i=1}^{k} w_i(y_{1,k}, \bar{s}_{k+1}, \dots, \bar{s}_n)$$

where W_k is defined in (1.14).

Definition 1.9 (Partial cooperative equilibrium)**.** A vector $s^{NC}(k) = (\bar{s}_{1,k}, \bar{s}_{k+1}, \ldots, \bar{s}_n) \in S$ satisfying the above Nash equilibrium requirements is called a *partial cooperative equilibrium under the Nash-Cournot assumption* of the game G where k players sign the agreement. The value $W_k(s^{NC}(k))$ is called the aggregate welfare of the signatories under the Nash-Cournot assumption and level of cooperation k.

Examples of partial cooperative games are in [BCK00], [MT08], [EF06], [DG08] and [MT12].

1.2.1 Dynamic Games

In this part we introduce dynamic games (for a complete discussion see [Lon10], [BO99] and [JZ99].)
We extend a game over a time horizon, that could be both finite or infinite. If the game is continuously time-dependent, we call it differential game. The time dependence of these games, brings some additional properties compared to static games. First of all, the overall payoffs for players are the sum (or integral) of the discounted payoffs over the time horizon. Moreover, the value of the payoff depends both on the actions of players and on the *state of the system*, which is represented by one or plus *state variable*.
In turn, the state of the system depends on the actions of the players, that we represent with *control variables*. Finally, the rate of change of a state variable is described by a difference or a differential equation. We consider dynamic games in the dynamic optimization framework. So, we define a N-player differential game as follows

Definition 1.10 (N-player differential game)**.** A N-player differential game of pre specified fixed duration, involves the following

1. A set I of players, $I = \{1, \ldots, N\}$.
2. A time interval $[0, T]$, which is specified a priori and denotes the duration of the game.
3. An infinite set S with some topological structure, called the trajectory space of the game. Its elements $\gamma(t)$, $0 \leq t \leq T$, represent the possible state trajectories of the game. For each fixed $t \in [0, T]$, $\gamma(t) \in S_0$, where S_0 is a subset of $\mathbb{R}^n$.

4. An infinite set U with some topological structure that we define control space, whose elements $u(t)$ are called control functions.

5. A differential equation

$$\dot{\gamma}(t) = f(t, \gamma, u), \qquad \gamma(0) = \gamma_0,$$

which describes the state trajectory of the game for a control u and a given initial state γ_0.

6. A payoff function, correlated with a functional $J(\cdot)$

$$J(u) = \int_0^T L(s, \gamma, u) ds + \phi(\gamma(T)).$$

It's always possible to consider a non-cooperative game as an optimization problem, than finding the stationary points of a given problem is equivalent to find fixed points in best reply correspondence of the game (for a proof see [Fin01]). Consider a game in which players can control the state of a given system, described by the trajectory function $\gamma(t)$, through a control function $u(t)$. That's, $\gamma(\cdot)$ solves the dynamical system

$$\dot{\gamma}(t) = f(t, \gamma, u), \qquad \gamma(0) = \gamma_0.$$

To find the Nash equilibrium of that game, we need to introduce a payoff functional $J(\cdot)$, as follows

$$J(u) = \int_0^T L(s, \gamma, u) ds + \phi(\gamma(T)).$$

Than, suppose that $U \subset \mathbb{R}^n$ is a closed set, $f \in C([0, T] \times \mathbb{R}^n \times U)$ satisfies the conditions (1.2) and that $L : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ are continuos and bounded from below functions. Finding the Nash equilibrium of a differential game, is equivalent to solve the Optimal Control problem

$$\inf_{u(\cdot)} J(u) = \int_0^T L(t, x^{x_0}(t), u(t)) dt + \phi_T(x^{x_0}(T)),$$

where $u \in C_\star([0, T], U)$ and $x^{x_0}(\cdot, u) \in C^1_\star([0, T], \mathbb{R}^n)$ is a solution of

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(0) = x_0. \end{cases}$$

From Dynamic Optimization theory, we have two possible approaches to that kind of problem, than we have two kinds of strategies and controls. We distinguish between *open-loop strategies* and *Markov-perfect strategies* (or feedback strategies). The first ones, are available when a player can make a credible pre-commitment, so an open-loop strategy is a planned time path of actions chosen by a player. That follows, a open-loop Nash equilibrium (OLNE) is a profile of open-loop strategies, such that each player's open-loop strategy maximizes its payoff, given the open-loop strategies of other players. OLNEs are time consistent, but not robust to perturbations: that's to say, along the equilibrium path, no one as an incentive to deviate, but if someone do this (also for an error), than it will be no optimal for other players to continue on original plan. Clearly, the credibility of the commitment is a focus point in open-loop strategies: each player can be suspicious about the real intentions of the other players, and the moral hazard problem rises.

On the other side, a Markov-perfect strategy is a rule that governs actions at any date on the observation of the state variables at that date. Consequently, a Markov-perfect Nash equilibrium (MPNE) is a profile of Markov-perfect strategies, such that for every couple $(t, \gamma(t))$, where t represent time and $\gamma(t)$ the state of the system at time t, the objective function of every player is maximized, given the Markov-perfect strategies of the other players. So, if agents use a path strategy, they make a binding commitment at the starting point about the actions they will take in the rest of the game. Otherwise, if they use a *decision rule*, the actions at any date t, will be a consequence of the observed state of the system at that date. Nevertheless, a Nash equilibrium in decision rules case, is not necessary a MPNE. To be Markov-perfect it must satisfy the condition that the continuation of the given decision rules is also a Nash equilibrium when viewed by any future $(t, \gamma(t))$ pair. The search for a MPNE seems to be a better way forward than a OLNE, and in facts is the most used method. The issue, however, is not so easy to deal with. The dependence of the equilibrium by the state of the system in every time, don't eliminates the moral hazard, to the extent that a player can manipulate the actions of the others, influencing the state variables. Thus, MPNE could be a better concept for equilibrium if we assume that players are sophisticated and manipulative. On the other side, for a several kind of games, the OLNE gives an higher payoff to players than the MPNE. So, if players have the ability to commit, an agreements to play the open-loop strategy could be incentivized. Moreover, an OLNE is easier to compute than MPNE, and this could be a great advantage, especially in such game in which Markov-perfect strategies gives a small gain in sophistication. In the end, the choice between OLNE and MPNE is based on the assumption about the ability to pre-commit. In the first, players pre-commit the whole time path of actions, in the latter players are not able to pre-commit at all. Clearly, there are middle ways, so in some cases players are able to pre-commit in the short run, but not in the long run (see [RS85]).

Applications to Economics are in several fields of the discipline, like management science (see e.g., [JMZ10], [BG12], [VZ09]), environmental protection (see e.g., [RC05], [RU07], [BL98]), or industrial organization (see e.g., [FK87] and [JV04]). Let us conclude this section with a game proposed in [Lon92], about transboundary pollution in infinite time horizon.

Example 1.5. *Consider a world with only two countries. For each country i, we denote with $y_i(t)$ the output at time t and we assume that output is proportional, with factor normalized at one, at the emission $E_i(t)$, than $E_i(t) = y_i(t) \quad \forall t$.*

The stock of pollutant at time t, common to both countries, is denoted by $S(t)$ and follows the differential equation

$$\dot{S}(t) = E_1(t) + E_2(t) - \delta S(t), \tag{1.15}$$

where $\delta > 0$ is the natural rate of decay. Moreover, we assume both quadratic functions, for utility, $B_i(t)$, and for damage, $D_i(S(t))$

$$\begin{cases} B_i(t) = a_i y_i(t) - \frac{1}{2}(y_i(t))^2, \\ D_i(t) = \frac{c_i}{2}(S(t))^2, \end{cases}$$

where a_i and c_i are stritcly positive parameters.

So, every country objective is maximize their own social welfare, w_i, subject to equation (1.15)

$$w_i = \int_0^\infty [B_i(t) - D_i(S(t))]e^{-\rho t} dt,$$

where $\rho > 0$ is the rate of discount.

Let us begin with the research of the OLNE.

Assume that country i believes that the other country j plays an open-loop strategy $E_j(t) = K_j^{OL}(t)$, so the maximization problem is

$$\max_{E_i(\cdot)} \int_0^\infty \left[a_i E_i(t) - \frac{1}{2}(E_i(t))^2 - \frac{c_i}{2}(S(t))^2 \right] e^{-\rho t} dt, \tag{1.16}$$

subject to

$$\dot{S}(t) = E_i(t) + K_j^{OL}(t) - \delta S(t), \qquad S(0) = S_0.$$

This is an optimal control problem, and we can use the Maximum Principle to solve it. So, consider the Hamiltonian function of the problem, H_i

$$H_i = a_i E_i - \frac{1}{2}(E_i)^2 - \frac{c_i}{2}(S)^2 + p_i(E_i + K_j^{OL} - \delta S),$$

where p_i *is the costate variable. For Pontryagin Principle, we have three necessary conditions, that are*

$$\frac{\partial H_i}{\partial E_i} = a_i - E_i + p_i = 0, \tag{1.17}$$

$$-(\dot{p}_i - \rho p_i) = \frac{\partial H_i}{\partial S} = -c_i S - p_i \delta, \tag{1.18}$$

$$\dot{S} = \frac{\partial H_i}{\partial p_i} = E_i + K_j^{OL} - \delta S, \tag{1.19}$$

with the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} p_i(t) S(t) = 0. \tag{1.20}$$

We can rearrenge (1.17) *respect* p_i *and reduce the system by this way*

$$-(\dot{E}_i - \rho(E_i - a_i)) = -c_i S - \delta(E_i - a_i),$$
$$\dot{S} = E_i + K_j^{OL} - \delta S, \qquad S(0) = S_0,$$
$$\lim_{t\to\infty} e^{-\rho t}(E_i(t) - a_i)S(t) = 0.$$

Clearly, for player j *we have a similar set of equations. The point is: to find an OLNE, we must find a couple of open-loop strategies* (K_1^{OL}, K_2^{OL})*, such that* $K_1^{OL}(t) = E_1^\star(t)$ *and* $K_2^{OL}(t) = E_2^\star(t)$*, where* $(E_1^\star, E_2^\star, S^\star)$ *solve the three differential equations*

$$\dot{E}_1(t) = c_1 S(t) + (\rho + \delta)(E_1(t) - a_1),$$
$$\dot{E}_2(t) = c_2 S(t) + (\rho + \delta)(E_2(t) - a_2),$$
$$\dot{S}(t) = E_1(t) + E_2(t) - \delta S t, \qquad S(0) = S_0,$$

with the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}(E_1(t) - a_1)S(t) = 0,$$
$$\lim_{t\to\infty} e^{-\rho t}(E_2(t) - a_2)S(t) = 0.$$

The solution of the system $(E_1^\star, E_2^\star, S^\star)$ *is unique and converges to a unique steady state* $(\hat{E}_1, \hat{E}_2, \hat{S})$*, as follows*

$$\begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{S} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} -a_1\delta(\rho+\delta)^2 - a_1c_2(\rho+\delta) + a_2c_1(\rho+\delta) \\ -a_2\delta(\rho+\delta)^2 - a_2c_1(\rho+\delta) + a_1c_2(\rho+\delta) \\ -(a_1+a_2)(\rho+\delta)^2 \end{bmatrix},$$

where J *is the Jacobian matrix*

$$\begin{bmatrix} \rho+\delta & 0 & c_1 \\ 0 & \rho+\delta & c_2 \\ 1 & 1 & -\delta \end{bmatrix}.$$

Finally, we consider the case of Markov-perfect strategies.
The difference respect the open-loop strategies, is that the country i *assumes now that country* j *uses a feedback emissions strategy* $E_j(t) = K_j^{FB}(S(t))$*. Thus, the optimal control problem is now*

$$\max_{E_i(\cdot)} \int_0^\infty \left[a_iE_i(t) - \frac{1}{2}(E_i(t))^2 - \frac{c_i}{2}(S(t))^2 \right] e^{-\rho t} dt, \tag{1.21}$$

subject to

$$\dot{S}(t) = E_i(t) + K_j^{FB}(S(t)) - \delta S(t), \qquad S(0) = S_0.$$

To solve the problem, the most useful method is dynamic programming. So, called $v_i(\cdot)$ *the value function of player* i*, we have the HJB equation*

$$\rho v_i(S) = \max_{E_i} \left[a_iE_i - \frac{1}{2}(E_i)^2 - \frac{c_i}{2}(S)^2 + v_i'(S)(E_i + E_j(S) - \delta S) \right]. \tag{1.22}$$

Moreover, we impose that v *don't increase too fast along the equilibrium path:*

$$\lim_{t\to\infty} e^{\rho t} v(S(t)) = 0.$$

The first order condition of equation (1.22) gives us an expression for $E_i(S)$:

$$E_i(S) = a_i + v_i'(S).$$

To simplify the calculation, we assume that the two countries are symmetric, so they have the same parameters: $a_i = a_j = a$ and $c_i = c_j = c$. As a consequence we have $v_i'(S) = v_j'(S) = v'(S)$, than $E_i(S) = E_j(S) = E(S)$.
Substituting inside (1.22), we obtain

$$\rho v(S) = \frac{1}{2}[a^2 + 4av' + 3(v')^2] - \delta S v' - \frac{c}{2}S^2. \tag{1.23}$$

Let us use a quadratic guess for value function:

$$v(S) = -\frac{\alpha S^2}{2} - \beta S - \mu,$$

so, the emission function become

$$E(S) = a - \beta - \alpha S.$$

In order to determine the parameters α, β and μ, we substitute the expression of $v(S)$ and $v'(S)$ inside (1.23). What we found is a quadratic equation in S, of the form

$$p_0 + p_1 S + p_2 S^2.$$

Since this expression must hold for all S, parameters must satisfy the conditions

$$p_0 = 0, \qquad p_1 = 0, \qquad p_2 = 0$$

From these three conditions we can derive the parameters we need

$$\alpha = \frac{1}{3}\left[-\left(\delta + \frac{\rho}{2}\right) + \sqrt{\left(\delta + \frac{\rho}{2}\right)^2 + 3c}\right],$$

$$\beta = \frac{2a\alpha}{\rho + \delta + 3\alpha},$$

$$\mu = \frac{a - \beta}{2\rho}(3\alpha - \delta - \rho).$$

Finally, we have that S converge to the steady state

$$\hat{S}^M = \frac{2a(\delta + \rho + \alpha)}{(\delta + \rho + 2\alpha)(2\alpha + \delta)}.$$

With some algebra, it's easy to determine that $\hat{S} < \hat{S}^M$, and that the initial emission under the MPNE is also higher than that under the OLNE.

1.3 International Environmental Agreements

The last example is a problem of transboundary pollution control. We consider only two countries and we show only the case of non-cooperative solutions, that's, each player maximize his own welfare, which is given by the difference between benefit from each player's emissions and the damage-cost due to global emission[2]. But, the global nature of the problem, obviously leads us to think about what happens if the countries concerned cooperate.

International Environmental Agreements are a great issue for politicians and economists in the last three decades. One of the first examples is the Montreal Protocol on Substances that Deplete the Ozone Layer, in 1987. The protocol is an international treaty designed to protect the ozone layer by phasing out the production of numerous substances that are responsible for ozone depletion, like chlorofluorocarbons (CFCs), hydrochlorofluorocarbons (HCFCs) and hydrofluorocarbons (HFCs). The agreement has undergone eight revisions: London (1990), Nairobi (1991), Copenaghen (1992), Bangkok (1993), Vienna (1995), Montreal (1997), Beijing (1999) and Montreal (2007), and has been ratified by 197 countries. Another important agreement is the Kyoto Protocol, signed in the Japanese city of Kyoto in 1997. The treaty, which have seen some important defection during the years, commits State Parties to reduce Green House Gases (GHGs) emissions, with the objective to contain the global warming. To do this, the treaty created a market in which countries can exchange the right to emit.

So, if we assume at least two of the countries can join in a coalition, than the determination of Nash equilibrium is only a part of the problem. In fact, now, another critical point is: how can it be built a stable coalition? In a non-cooperative approach, there's not a supranational authority that can force countries to cooperate, so the focus is on mechanism design of the agreement. These games, both static and dynamic, are two stage games: in the first stage, called *membership game*, players decide whether join or not in coaltion, while in the second one, called *emission game*, players establish their emissions, maximizing the joint welfare if they are in coalition or their single welfare if they stay outside.

It's clear that in non-cooperative context, the well known problem of free-riding is inevitable. In the case of common resources, a free-rider is a player who can benefits from these resources without pay for them. So, when we think to membership game we need

[2]For a discussion about this kind of aggregative games see [MT09].

to define some conditions to disincentive the free-riders.

The conditions commonly used in literature are those developed in [dAs+83], in which some conditions for a self-enforcing agreement are defined in the case of a cartel. Consider a n-player game, and assume that k join in a coalition and the other $(n-k)$ act non-cooperatively. We denote with $w_i(k)$ the welfare of i-th player when the coalition is formed by k player. The first condition is called *internal stability* condition: a coalition is stable if its dimension k is such that for every player i inside the coalition holds

$$w_i^C(k) \geq w_i^{NC}(k-1), \tag{1.24}$$

where C denote cooperative welfare and NC denote the non-cooperative one. This condition can be interpreted as follows: k is the dimension of a stable coalition if the welfare of a cooperative player when coalition is formed by k players is at least equal to its non-cooperative welfare when coalition is formed by $(k-1)$ players. In other word a coalition is internally stable if there is not a player inside who has an incentive to defect.

The second condition is called *external stability* condition: for every player j outside coalition, k must verify

$$w_j^{NC}(k) \geq w_j^C(k+1). \tag{1.25}$$

That's to say, no one of the $(n-k)$ players outside coalition as an incentive to join the agreement. Of course, these condition are the most used in literature, but are not the only: for a different approach to the stability problem see [Fin03] and [EF06].

There is a very extensive literature on International Environmental Agreements (IEAs), that starts in early nineties with the static models proposed by Carraro and Siniscalco (see [CS93]) and Barrett (see [Bar94]). A brief analysis of these models allow us to show some general issues on IEAs problem.

Carraro and Siniscalco (1993) [CS93]. They consider a world with n symmetric countries $(n \geq 2)$, where each country benefits from using the environment as a factor of production and as a receptacle of emissions. Nevertheless, this exploitation of natural resources generate a damage, via emissions. If we denote with x_i the emissions of country i, and with x_{-i} the vector of emissions of the other $(n-1)$ countries, we can represent the welfare $P_i(x)$ of country i as follows

$$P_i(x) = f_i(x_i) - D_i(x_i, x_{-i}),$$

where $f_i(x_i)$ is the benefit function, $D_i(x_i, x_{-i})$ is the damage function and $x = (x_i, x_{-i})$. The benefit function is related to the use of environment for production and consumption, and depends on abatement costs, that's to say that depends on the level of technologies, on the economic structure, on the general level of development and the endowment of resources of a country [3]. Inside the word *technology*, there are not only the industrial processes who turn inputs in outputs, but we mean also the useful knowledge and experience, institutions, organizational structure, norms and values who govern the production processes. On the other side, the damage function $D_i(x_i, x_{-i})$ depends on the perception that country i has about the negative effects of emissions of pollutant and on the evaluation of such effects. Then, the damage function is a subjective evaluation of environmental goods. These games are two stage games: really the membership game is a *metagame*, because of each player choosing whether join or not in coalition, anticipates the choice of the other players and the relative outcomes in terms of emission levels. So, the solution of emission game is clear: non-cooperative players maximize their own welfare, that's to say that they choose the emission corresponding to a fixed point in best reply function, while cooperators choose the emissions that maximize the joint welfare. The point is: what is the solution of membership game?

The dimension of a stable coalition is the integer number k, who verifies the inequalities (1.24) and (1.25). But, assume that the grand coalition, that is when all players join the agreement, is the social optimum. Then, the real question is: how can we design the agreements to have a stable coalition with $k = n$ players?

Carraro and Siniscalco show that in a static context with symmetric player the environment problem not necessary leads to the *tragedy of commons*. Indeed, there is a range of possible voluntary cooperators, who join in a profitable and stable agreement for pollution control. Moreover, it's possible to expand the size of the coalition if members decide to transfer the gains of cooperation to non-members. Nevertheless, the only forecast of a transfer scheme is not enough to sustain a large coalition, that require a certain degree of commitment by the players.

Barrett's model (1994) [Bar94]. It is a game with n symmetric players, where the

[3]Henceforth we will use indifferently the terms benefit function and production function for $f_i(\cdot)$.

strategic variables are not emissions, but abatement levels. There is a duality between the two kinds of problems, so results on cooperation level are valid in both cases (for a discussion see [Fin01] and [DS06]). So, he consider for each player i a benefit function, $f_i(\cdot)$, who depends on the global abatement level, as follows

$$f_i(Q) = \frac{b\left(aQ - \frac{Q^2}{2}\right)}{N},$$

where a and b are positive parameters, N is the total number of players and $Q = \sum_{j=1}^{N} q_j$ is the total abatement level, with q_j is the abatement level of each player j. On the other hand, each player i has to incur a cost for abatement, denoted by $D_i(\cdot)$

$$D_i(q_i) = \frac{cq_i^2}{2},$$

where c is a positive parameter. In this context, cooperators can punish a country that withdraws from the agreement reducing their abatement level, and the threat is credible because of the signatories always maximize their joint welfare. Nevertheless, Barrett shows, by numerical simulations, that this is not sufficient to have a stable and large coalition. In fact, or a self-enforcing IEA do not exist, or it consist of no more than two or three countries. The only way to have a coalition that is stable and large is when the gain from cooperation respect non-cooperation close to zero. Moreover, the results are the same considering a linear benefit function, $f_i(Q) = \omega Q$, but, in this case is possible to produce an analytical proof. So, denote by α the fraction of players that join the agreement, the optimal solution are

$$q_{ns} = \frac{\omega}{c} \quad \text{and} \quad q_s = \frac{\omega \alpha N}{c},$$

where ns denote the non-signatories and s denote the signatories. In order to obtain the stability condition we need four expressions for welfare, that are, substituting optimal emissions

$$w^{ns}(\alpha) = \frac{\omega^2 N(1-\alpha+\alpha^2 N - \frac{1}{2N})}{c}, \tag{1.26}$$

$$w^{s}(\alpha) = \frac{\omega^2 N(1-\alpha+\frac{\alpha^2 N}{2})}{c}, \tag{1.27}$$

$$w^{ns}(\alpha - 1/N) = \frac{\omega^2 N(1-3\alpha+\frac{3}{2N}+\alpha^2 N)}{c}, \tag{1.28}$$

$$w^{s}(\alpha + 1/N) = \frac{\omega^2 N(1-\frac{1}{2N}+\frac{\alpha^2 N}{2})}{c}. \tag{1.29}$$

Now, from the internal stability, given by $w^s(\alpha) - w^{ns}(\alpha - 1/N) \geq 0$, we obtain the solution $1/N \leq \alpha \leq 3/N$. If we consider the external stability, $w^{ns}(\alpha) - w^s(\alpha + 1/N) \geq 0$, we found the solution $\alpha \geq 2/N$. Follows that the system is verified when $\alpha N \in [2, 3]$, so if $N = 2$ we have that both countries sign the agreement, instead when $N \geq 3$ the self-enforcing agreement consists of three countries.

These two models allow us to show some critical points. First, without detracting from the validity of the arguments, it's clear that pollution control is in reality a dynamic problem. Static models can give important contributions to discussion on stability IEAs, but dynamical models give a more realistic picture of the phenomenon. Second, as we saw, the treatment of stability is not really easy in analytical way, so, for sake of simplicity, a large part of the literature consider frameworks with symmetric countries. Than, the introduction of a certain degree of asymmetry between countries, give for sure a contribution to descriptive power. Third, what emerges from the models, is that if we want a stable and large coalition, if possible the grand coalition, we need to create some mechanisms that incentive players to cooperate, or that disincentive players to non cooperate. All these issues have been addressed in the literature, and there is a very extensive stream of works who approach the problem in dynamic case.

In the last years various models consider a world divided at least in two kinds of players, developed and developing countries. Clearly, the most realistic way is to consider that each player has its own parameters, but for sake of simplicity, the division in developed and developing countries is a good approximation of the principle factor of asymmetry. For the last problem, several proposals to design a self-enforcing agreements, both static and dynamic has been presented. For example, we have models based on IEAs supported by trigger strategies, that's to say that if a player in coalition defects, the other cooperators can punish him. Or, as we saw with the model of Carraro and Siniscalco, the gain from cooperation can be used, via transfer scheme, to enlarge a stable coalition. Finally, we have the way known as *issue linkage*, that's try to make the agreement economically advantageous linking the IEAs to another agreement, that could be R&D, or trade or another economic issue [4].

Pavlova and de Zeeuw (2013) ([PZ13]). They consider a world with N asymmetric countries. Each country i emits a pollutant e_i, and receives a benefit from production, defined by the function $f_i(\cdot)$

$$f_i(e_i) = \delta_i \left(\alpha_i e_i - \frac{1}{2} e_i^2 \right),$$

where α_i and δ_i are strictly positive parameters. On the other side, global emissions found a damage to each player i:

$$D_i(S) = \beta_i S,$$

where $\beta_i > 0$, is a parameter, and $S = \sum_{j=1}^{N} e_j$ is the global emission. As usual, we suppose that k players join the agreement, while the others, $(N-k)$, act non cooperatively. From optimization problems we can derive the emissions, as follows

$$e_i^C = \alpha_i - \frac{\sum_{h=1}^{k} \beta_h}{\delta_i} \qquad \text{and} \qquad e_j^{NC} = \alpha_j - \frac{\beta_j}{\delta_j},$$

for each player i in coalition, and for each player j outside coalition. Suppose now, that we have only two kinds of countries, than $N = N_1 + N_2$. So, players of kind 1 are identified by parameters $(\alpha_1, \delta_1, \beta_1)$ and players of kind 2 by parameters $(\alpha_2, \delta_2, \beta_2)$. For the analysis of stability, we can use the conditions (1.24) and (1.25) to find the maximal size and the composition of the coalition. Clearly, in the case of asymmetric players the stability conditions must hold both for players of kind 1 and kind 2. First, we note that in case of symmetry the maximal size of a stable coalition is three, that's a standard results in literature. If we consider two kinds of countries, we can summarize the results in the following proposition.

[4] for a different and interesting approach see [MRS14]

Proposition 1.11. *The maximal size of a stable coalition consists of two countries of kind 1, and N_2 countries of kind 2, provided that $\beta := \frac{\beta_1}{\beta 2}$ is large enough and $\delta := \frac{\delta_1}{\delta 2}$ is small enough.*

Then, the maximal size of a stable coalition is that one composed by two players ok kind 1 and all players of kind 2, if some conditions on parameters are verified. This conditions require that the marginal damage from pollution of players of kind 1 is greater enough respect marginal damage of players of kind 2, while their shifting marginal benefits are very closer.

Recently, some works have addressed the problem in a dynamical way, like [RC05], [RU07], [Zee05], [Bah+09] and [BSZ10]. Rubio and Casino, provide a N-player differential game with symmetric countries, who are characterized by quadratic production and damage-cost functions. The main result of this model is that, in dynamic context the maximum size of a stable agreement is only two players, independently from the gain of cooperation and from the use of open-loop or feedback strategies. [Bah+09] introduces asymmetric countries, and uses the calibration from MERGE (model for evaluating regional and global Green House Gases reduction) to establish in which conditions is possible to achieve the full cooperation. In [BSZ10] were considered N asymmetric players, and that a fraction s of their join the agreement, while the rest $(1-s)$ defects. In this model the analysis of stability don't deals with the self-enforcing conditions (1.24) and (1.25), but is assumed that s follows an evolutionary dynamics based on imitation of the best. What they found is that when punishment has a cost, it's possible to have a solution with no countries in coalition, along with partial or full cooperation. The final result depends on the initial conditions: if the initial coalition is not large enough for a given value of stock of pollutant, than the equilibrium solution is full defection. Partial or full cooperation can be reached increasing the punishment or decreasing the cost for punishing. In case that the punishment is without cost, full defection can't be observed, and the partial or full cooperation don't depends by initial conditions, but only on the parameters value.

We said that the stability of a large coalition depends on how the agreements are drawn. We said, moreover, that the most used mechanisms are trigger strategies, transfer schemes and issue linkage. We want to go a different way, following the idea of Social Externality presented in [CR06].

Cabon-Dhersin and Ramani (2006) [CR06]. In their static model, they consider a world with N symmetric countries, k of which decide to choose jointly their abatement levels, and $(N-k)$ maximize their own welfare. The functional forms of the model consider a linear benefit function, $f_i(\cdot)$, and a quadratic damage-cost function, $D_i(\cdot)$, for each player i

$$f_i(Q) = \omega Q, \qquad D_i(q_i) = \frac{cq_i^2}{2},$$

where ω and c are strictly positive parameters, q_i is the abatement level of player i and $Q = \sum_{j=1}^{N} q_j$ is the global abatement. The assumption now, is that the welfare of cooperators is affected by a social externality, that's a positive function of the number of players and not depends on abatement levels. So, if we denote with q^C the optimal abatement levels for cooperators and with q^{NC} the optimal abatement levels for non cooperators, than the welfare for each player i in coalition is

$$w^C = \omega(kq^C + (N-k)q^{NC}) - \frac{cq^C}{2} + sk,$$

where $s > 0$ is a parameter. On the other side, the welfare for each non cooperator is

$$w^{NC} = \omega(kq^C + (N-k)q^{NC}) - \frac{cq^{NC}}{2}.$$

From optimization we obtain the two expressions for abatement levels

$$q^C = \frac{\omega k}{c}, \qquad q^{NC} = \frac{\omega}{c}.$$

By stability conditions we can determine the interval in which falls the size of the stable coalition: $k \in [k^\star - 1, k^\star]$, where $k^\star = \tilde{s}/2 + 2 + \sqrt{\tilde{s}^2/4 + 2\tilde{s} + 1}$ and $\tilde{s} = 2\frac{c}{\omega^2}s$. It's clear that k is an increasing function of $\tilde{s}$, so for a suitable choice of c, ω, and s, we can have a stable grand coalition. Let us explain our interpretation of this Social Externality.
From a mathematical point we assume that a strictly positive function, denoted by $Ext(\cdot)$ is added to welfare of every cooperator, and this function depends only on the number of players in coalition. For this reason, $Ext(\cdot)$ has no effects on the maximization process. There is a stream in literature that studies the possibilities of optimality of multiple coalition (see [Fin03]) and this issue was proposed in the lasts Conferences of Parties (COP), so it's also a political question. Nevertheless, we assume that the grand coalition is the Social Optimum, that's a classical hypothesis in literature. So, we need to improve some mechanisms that help to obtain this final result.

Basically, the point is to create some incentives to join or some disincentives to defect, and we spoken about trigger strategies, transfer scheme and issue linkage. Our choice to try a different way, rise from some practical considerations.
The first one, is that it's now necessary involve development countries in emission's reduction process. There are some of this countries, like China, India, Brazil and South Africa (that with Russia formed the so-called BRICS), that contribute significantly to pollution. But, following the Kyoto Protocol, is very difficult to ask these countries to implement pollution control policies. For historical reasons, because they have not responsibility about the level of pollution and for economics reason. The fact is that environment is a "luxury" good for these countries, because their principal issue is to increase the wealth per capita, build infrastructure, increase the level of instruction, etc. So, from one hand there's that is hard to think that developing countries act for some kind of farsightedness and, on the other side, they have too economic power to imagine that a punishment is a credible threat or that anyone would pay for their collaboration. The issue linkage is the closer idea to our approach, but we think that specifying one kind of collateral agreement there is a significant loss in possible incentives. So, we introduce in our model the concept of Social Externality, like in [CR06]. The idea is: when players have to decide whether join the coalition or not, they consider the possibility to earn an extra-payoff, not related to pollution control, just for the reason that joining creates a connection with other countries. Classical example is Russia, that ratified the Kyoto protocol with the hope to have more consideration when its entry in World Trade Organization (WTO) would have been voted. We know that is very vague, because potentially within this concept of externality are all the possible relations that countries could establish, and this could bring a loss in descriptive power. On the other side, we think that this loss it's acceptable, consider the great flexibility that the vagueness of the externality create for the design of the agreement. So, the purpose of this Thesis is verify if the Social Externality could be a good mechanism to bring the grand coalition. We consider a Global Emission Game (GEG) and a world with asymmetric countries, where a GEG is defined as follows

$$GEG = \{I, \Delta, f_i(e_i), D_i(S), w_i, S\},$$

where $I = \{1, 2..., n\}$ is a set of n asymmetric players, Δ is the set of emission strategies, $f_i(e_i)$ is the benefit function from emissions for a player i, $D_i(S)$ is the damage-cost

function from stock of pollutant, w_i is the net benefit, given by $f_i - D_i$, S is the stock of pollutant.
Specifically, we assume that the world is divided in two kinds of countries, developed and developing countries, and that there is homogeneity within two groups. We deal with three different games, one static and two differential, where a Social Externality term is added to agent's welfare function. In the static case the Social Externality allows to obtain new results that generalize the work of [CR06] because we deal with asymmetric countries.

In our work we include the concept of Social Externality in the differential context, considering first a two player game and then extending the model to the case of N-player game. We consider the two-player differential game on pollution control in [MZ13] where the damage cost function takes into account a gradual involvement approach for environmental problems of the developing country: this model is extended to a N-player situation and adding the Social Externality effect allows to enlarge the coalition of cooperating countries.

Chapter 2

N-player Static Game

In this Chapter we investigate the question if a Social Externality leads to a stable coalition in a N-player static game, in the spirit of Cabon-Dhersin and Ramani ([CR06]). We deal with a game with N asymmetric players by considering the membership game and the global emission game. As usual, first we will face the emission game and find the Nash equilibrium in a partial cooperative framework.
Thereafter, we will proceed with the membership game and find some stability conditions considering a self-enforcing agreement. In order to give an analytical proof of the existence of a stable coalition, we assume that countries are divided in two homogeneous groups, developed and developing countries. This assumption is in line with the arrangement of the Kyoto Protocol, which have been provided different commitments for developed countries and for countries in the developing.
To conclude we present some numerical results about the size and the composition of a stable coalition. The Chapter is divided as follows: in Section 1 we present the assumptions of the model and its functional forms; in Section 2 we characterize the emission solutions and the expressions of the optimal welfares; in Section 3 we show our results about the membership game.

2.1 The Model

In this section we present the model. We consider a world with N asymmetric countries that correspond to our set of players I, where $I = \{1, ..., N\}$. We make some commonly assumptions about the functional forms of the model.
Let us remind that we are modeling a global emission game, that we define in normal form, and for a static game, as follows.

Definition 2.1. We call Global Emission Game (GEG) a normal form game

$$GEG = (I, S, w_i),$$

where $I = \{1, 2..., N\}$ is a set of N asymmetric players, S (S is a real interval) is the set of emission strategies common to all players and w_i is the payoff of each player i. We denote with $f_i : S \to \mathbb{R}$ the production function of player i, where $e_i \in S$ describes the emissions of the player. The idea behind is that there is a direct correlation between production and pollution[1]. Moreover $D_i : \mathbb{R} \to \mathbb{R}$, defined by $D_i(E)$ is the damage-cost function for player i, where E is the global emission, $E = \sum_{p=1}^{N} e_p$. Than, the net benefit of player i from emissions is given by

$$w_i(e_1, \ldots, e_N) = f_i(e_i) - D_i(E),$$

We lie within the partial cooperative games literature ([MT08], [MT09], [CGL11] and [BCK00]), so we assume that k players join the coalition and $(N-k)$ players defect. Thus, we can write the maximization problems, both for non-cooperators and cooperators, as follows

$$\max_{e_j} w_j = \max_{e_j} \{f_j(e_j) - D_j(E)\},$$

where w_j is the welfare of a country j outside coalition. Without loss in generality, we suppose that the first k players in the set I join the coalition, so we denote by $C = \{1, \ldots, k\}$ the set of cooperators. They maximize the joint welfare and the optimization problem is

$$\max_{e_1,\ldots,e_k} \sum_{i=1}^{k} w_i = \max_{e_1,\ldots,e_k} \sum_{i=1}^{k} [f_i(e_i) - D_i(E)].$$

[1] Mathematically, we assume that emissions are by-product of industrial activities x_i, that's to say $e_i = g(x_i)$. If we take $g(\cdot)$ as a smooth function, we can invert it and write $x_i = g^{-1}(e_i) := f_i(e_i)$.

Due to the nature of the problem, we make the assumption that $e_i \in \Delta$ is positive for all $i \in I$, and that the strategy set for each player i, $\Delta \subset \mathbb{R}$, is compact.

Moreover, we make some typical assumptions on the functions that describe the model (see [Fin01]). First, we choose a twice continuously differentiable damage-cost function $D_i(E)$ such that: $D_i'(E) \geq 0$ and $D_i''(E) \geq 0$, $\forall e_i \geq 0$, and $D_i(0) = 0$. If $w_i'' < 0$, than $f_i''(e_i) < D_i''(E)$, $\forall e_i \geq 0$. Finally, the production function $f_i(e_i)$ is chosen such that: $f_i'(e_i) \geq 0$, for all $0 \leq e_i \leq \bar{e}_i$, while we don't make assumptions on $f_i'(e_i)$ $\forall e_i \geq \bar{e}_i$, and $e_i(0) = 0$. The assumptions on the first derivatives of $f_i(e_i)$, imply that there is an emission level $\bar{e}_i$, before which the benefits are an increasing function, while after $f_i(e_i)$ may either continue to increase or remain constant or decrease. The choice $f_i''(e_i) < 0$ reflects the standard hypothesis of decreasing marginal economies of scale in the production[2]. The assumptions on damage-cost function, $D_i(E)$, describe the fact that damage increase in emissions at an increasing rate. The interpretation is that when the rate of contamination is higher, than self-purification of environmental system is diminishing.

There might be an upper bound of global emissions, above which the system collapse, but we assume that this level is higher than the global emissions which derive from maximization of welfares. Another way to interpret the assumption $D_i''(E) \geq 0$ is that it could be viewed as society's willingness to pay for emission reduction. So, when the global emissions rise to very high levels, this willingness increase more than proportionally respect emissions.

2.1.1 Functional Forms

We present in the following the functional forms of the model by choosing the production, the damage-cost and the externality functions.

So, for each player i, we consider

$$\begin{aligned}
f_i(e_i) =& \delta_i \left(\alpha_i e_i - \frac{1}{2} e_i^2 \right), \\
D_i(e_i) =& \beta_i \left(\sum_{i=1}^{N} e_i \right),
\end{aligned}$$

where, e_i is the emission of player i, and the parameters α_i, β_i (vulnerability to environmental damage) and δ_i (shifting marginal benefits) are strictly positive. We consider a

[2] Or decreasing marginal utility, if we consider emission like a consumption good.

quadratic production function that is a classic choice in literature, and a linear damage-cost function, that is not uncommon in literature (see e.g., [HS97], [BSZ10] and [PZ13]) and supported by some empirical estimations (see [LL03]). Nevertheless, the differences between a linear damage-cost function and a more realistic quadratic function (see e.g., [MRS14], [FP13] and [McG07]) are almost all quantitative, but not qualitative.
Denoting with Ext_i the externality, we assume a constant contribution from each player

$$Ext_i = s_i,$$

where s_i is a strictly positive parameter.
Now, the k players that form the coalition maximizing the joint welfare, and the remaining $(N-k)$ players act by their own, maximizing their single welfare. So, we have the optimization problem, for each player j not in coalition, as follows:

$$\max_{e_j} w_j^{NC} = \max_{e_j}\left\{\delta_j\left(\alpha_j e_j - \frac{1}{2}e_j^2\right) - \beta_j\left(\sum_{i=1}^{n} e_i\right)\right\}. \tag{2.1}$$

For the coalition, we have a joint welfare maximization and moreover we hypothesize that a social externality, related only on number of players in coalition, affects the welfare. So the optimization problem is:

$$\max_{e_1 \dots e_k} \sum_{i=1}^{k} w_i^C = \max_{e_1 \dots e_k} \sum_{i=1}^{k}\left[\delta_i\left(\alpha_i e_i - \frac{1}{2}e_i^2\right) - \beta_i\left(\sum_{h=1}^{k} e_h + \sum_{h=k+1}^{n} e_h\right) + s_i\right]. \tag{2.2}$$

Definition 2.2. Any vector $(\bar{e}_1, \dots, \bar{e}_N) \in \Delta^n$ such that $\bar{e}_i$ satisfies (2.1) for all $i = k+1, \dots, N$ and $\bar{e}_i$ satisfies (2.2) for all $i = 1, \dots, k$, is called partial cooperative equilibrium.

We consider emissions as a by-product of industrial activities, and assuming that the function that relates emissions and production are smooth and invertible, we express the production for country i as a function of emission levels, and we indicate it with $f_i(e_i)$, being e_i the emission of country i. Moreover, our hypothesis is that developed countries have an higher degree of interest in environmental issues respect developing countries, both for economic and historical motivations. In the end, this is the same approach of the Kyoto protocol. The point is: a developing country need to improve his infrastructures, increasing per capita wealth, life expectance, instruction level, etc. In this context, environment is a "luxury good". In addition, the actual level of stock pollutant cannot be attributed to developing countries.

2.2 Emission Solutions and Welfares

In order to find the optimal emissions for the defectors, we solve the first order conditions of the problem (2.1). Deriving respect e_j and imposing the result equal to zero, we have

$$\frac{\partial w_j^{NC}}{\partial e_j} = \alpha_j \delta_j - \delta_j e_j - \beta_j = 0.$$

Rearranging we can find the expression for non-cooperative emissions, as follows

$$e_j^{NC} = \alpha_j - \frac{\beta_j}{\delta_j}. \tag{2.3}$$

In a similar way, we calculate the partial derivative of the problem (2.2), respect e_i, where $i = 1, \ldots, k$, and find the emission of every player in coalition solving the k-dimensional system obtained imposing all the derivative equal to zero

$$\begin{cases} \alpha_1 \delta_1 - \delta_1 e_1 - \sum_{h=1}^{k} \beta_h = 0, \\ \vdots \\ \alpha_i \delta_i - \delta_i e_i - \sum_{h=1}^{k} \beta_h = 0, \\ \vdots \\ \alpha_k \delta_k - \delta_k e_k - \sum_{h=1}^{k} \beta_h = 0. \end{cases}$$

Than the emission for each player i in coalition is:

$$e_i^C = \alpha_i - \frac{1}{\delta_i} \sum_{h=1}^{k} \beta_h. \tag{2.4}$$

To find the welfares of cooperative and non-cooperative players, we have to compute $w_i(e_1^C, \ldots, e_k^C, e_{k+1}^{NC}, \ldots, e_N^{NC})$ for any $i \in I$, that we denote by $w_i^C(k)$ for $i \in \{1, \ldots, k\}$

and by $w_j^{NC}(k)$ for $j \in \{k+1, .., N\}$

$$\begin{cases} w_i^C(k) = & \delta_i\Bigg[\alpha_i\Bigg(\alpha_i - \frac{1}{\delta_i}\sum_{h=1}^k \beta_h\Bigg) - \frac{1}{2}\Bigg(\alpha_i - \frac{1}{\delta_i}\sum_{h=1}^k \beta_h\Bigg)^2\Bigg] + \\ & -\beta_i\Bigg[\sum_{p=1}^k\Bigg(\alpha_p - \frac{1}{\delta_p}\sum_{h=1}^k \beta_h\Bigg) + \sum_{p=k+1}^N\Bigg(\alpha_p - \frac{\beta_p}{\delta_p}\Bigg)\Bigg] + \sum_{h=1}^k s_h, \\ & \\ w_j^{NC}(k) = & \delta_j\Bigg[\alpha_j\Bigg(\alpha_j - \frac{\beta_j}{\delta_j}\Bigg) - \frac{1}{2}\Bigg(\alpha_j - \frac{\beta_j}{\delta_j}\Bigg)^2\Bigg] + \\ & -\beta_j\Bigg[\sum_{p=1}^k\Bigg(\alpha_p - \frac{1}{\delta_p}\sum_{h=1}^k \beta_h\Bigg) + \sum_{p=k+1}^N\Bigg(\alpha_p - \frac{\beta_p}{\delta_p}\Bigg)\Bigg], \end{cases}$$

in which i is a cooperative player, j is a defector and the coalition consists of k players.

2.3 Stability

To establish the number of players of a stable IEA we refer to notions of internal and external stability (see [dAs+83]). We want highlight that these conditions are more stringent and there are different papers that try to propose different ways to face the problem (see [Fin03] and [EF06]). The basic idea is that a coalition is stable if no one inside has an incentive to defect and no one outside has an incentive to join in. So, called w the pay-off of a player, a coalition of k players is stable if are verified:

$$w_i^C(k) \geq w_i^{NC}(k-1), \qquad w_j^{NC}(k) \geq w_j^C(k+1),$$

where i is a cooperator and j is a defector. First condition is called *internal* stability, while the second is called *external* stability.

Substituting optimal emissions in welfare functions, we can make explicit the stability conditions. So, consider first the internal stability condition, we have

$$\begin{cases} w_i^C(k) = & \delta_i\left[\alpha_i\left(\alpha_i - \frac{1}{\delta_i}\sum_{h=1}^k \beta_h\right) - \frac{1}{2}\left(\alpha_i - \frac{1}{\delta_i}\sum_{h=1}^k \beta_h\right)^2\right] + \\ & -\beta_i\left[\sum_{p=1}^k\left(\alpha_p - \frac{1}{\delta_p}\sum_{h=1}^k \beta_h\right) + \sum_{p=k+1}^N\left(\alpha_p - \frac{\beta_p}{\delta_p}\right)\right] + \sum_{h=1}^k s_h, \\ & \\ w_i^{NC}(k-1) = & \delta_i\left[\alpha_i\left(\alpha_i - \frac{\beta_i}{\delta_i}\right) - \frac{1}{2}\left(\alpha_i - \frac{\beta_i}{\delta_i}\right)^2\right] + \\ & -\beta_i\left[\sum_{p=1,p\neq i}^k\left(\alpha_p - \frac{1}{\delta_p}\sum_{h=1,h\neq i}^k \beta_h\right) + \sum_{p=k+1}^N\left(\alpha_p - \frac{\beta_p}{\delta_p}\right) + \alpha_i - \frac{\beta_i}{\delta_i}\right]. \end{cases}$$

The internal stability condition is given by the inequality $w_i^C(k) - w_i^{NC}(k-1) \geq 0$. So, simplifying and rearranging we have

$$w_i^C(k) - w_i^{NC}(k-1) = -\frac{1}{2\delta_i}\left[\left(\sum_{h=1}^k \beta_h\right)^2 - \beta_i^2\right] + \beta_i\left[\frac{1}{\delta_i}\sum_{h=1,h\neq i}^k \beta_j + \beta_i \sum_{h=1,h\neq i}^k \frac{1}{\delta_h}\right] + \sum_{h=1}^k s_h \geq 0,$$

which becomes

$$w_i^C(k) - w_i^{NC}(k-1) = -\frac{1}{2\delta_i}\left(\sum_{h=1,h\neq i}^k \beta_h\right)^2 + \beta_i^2\left(\sum_{h=1,h\neq i}^k \frac{1}{\delta_h}\right) + \sum_{h=1}^k s_h \geq 0 \tag{2.5}$$

Consider now the external stability condition, we have

$$\begin{cases} w_j^{NC}(k) = & \delta_j\left[\alpha_j\left(\alpha_j - \frac{\beta_j}{\delta_j}\right) - \frac{1}{2}\left(\alpha_j - \frac{\beta_j}{\delta_j}\right)^2\right] + \\ & -\beta_j\left[\sum_{p=1}^k\left(\alpha_p - \frac{1}{\delta_p}\sum_{h=1}^k \beta_h\right) + \sum_{p=k+1}^N\left(\alpha_p - \frac{\beta_p}{\delta_p}\right)\right], \\ & \\ w_j^C(k+1) = & \delta_j\left[\alpha_j\left(\alpha_j - \frac{1}{\delta_j}\left(\sum_{h=1}^k \beta_k + \beta_j\right)\right) - \frac{1}{2}\left(\alpha_j - \frac{1}{\delta_j}\left(\sum_{h=1}^k \beta_k + \beta_j\right)\right)^2\right] + \\ & -\beta_j\left[\sum_{p=1}^k\left(\alpha_p - \frac{1}{\delta_p}\left(\sum_{h=1}^k \beta_k + \beta_j\right)\right) + \sum_{p=k+1, p\neq j}^N\left(\alpha_p - \frac{\beta_p}{\delta_p}\right) + \right. \\ & \left. +\alpha_j - \frac{1}{\delta_j}\left(\sum_{h=1}^k \beta_k + \beta_j\right)\right] + \sum_{h=1}^k s_h + s_j. \end{cases}$$

The external stability gives a number k such that, for each player j outside coalition holds the inequality $w_j^{NC}(k) - w_j^C(k+1) \geq 0$.
After some algebra, we obtain

$$\begin{aligned} w_j^{NC}(k) - w_j^C(k+1) = & \frac{1}{2\delta_j}\left[\left(\sum_{h=1}^k \beta_h\right)^2 + 2\beta_j\sum_{h=1}^k \beta_h\right] + \\ & - \beta_j\left[\beta_j\sum_{h=1}^k \frac{1}{\delta_h} + \frac{1}{\delta_j}\sum_{h=1}^k \beta_h\right] - \sum_{h=1}^k s_h + s_j \geq 0, \end{aligned}$$

which becomes

$$w_j^{NC}(k) - w_j^C(k+1) = \frac{1}{2\delta_j}\left(\sum_{h=1}^k \beta_h\right)^2 - \beta_j^2\left(\sum_{h=1}^k \frac{1}{\delta_h}\right) - \sum_{h=1}^k s_h + s_j \geq 0 \tag{2.6}$$

We can observe immediately that the social externality facilitates the stability of a grand coalition. If we consider the effects of the externality on the stability conditions we observe that it helps to obtain internal stability, but, on the other side, it has a negative effect on external stability. That's to say, if the externality is large enough, we can have that every coalition is internal stable, while there is no size k for which we have external stability. So, the only stable coalition is the grand coalition. Suppose now that there exist two kinds of player, developed countries, identified by subscript 1, and developing countries, called 2. We assume that within each subgroup players are homogeneous, and that the coalition is arranged by n_1 players of kind 1 and n_2 of kind 2, with n_1 and n_2 positive integers such that $n_1 + n_2 \leq N$. So, we are assuming that within each group, all players have the same parameters: for countries of kind 1 we have

$(\alpha_1, \delta_1, \beta_1, s_1)$, while for countries of kind 2 we have $(\alpha_2, \delta_2, \beta_2, s_2)$.

We must specify the stability conditions for developed countries and for developing ones. So, every condition generate two different inequalities.

For internal stability, we have the two inequalities

$$
\begin{aligned}
w_1^C(k) - w_1^{NC}(k-1) = & -\frac{1}{2\delta_1}\left((n_1-1)\beta_1 + n_2\beta_2\right)^2 + \\
& + \beta_1^2\left(\frac{n_1-1}{\delta_1} + \frac{n_2}{\delta_2}\right) + s_1 n_1 + s_2 n_2 \geq 0; \qquad (2.7a)
\end{aligned}
$$

$$
\begin{aligned}
w_2^C(k) - w_2^{NC}(k-1) = & -\frac{1}{2\delta_2}\left(n_1\beta_1 + (n_2-1)\beta_2\right)^2 + \\
& + \beta_2^2\left(\frac{n_1}{\delta_1} + \frac{n_2-1}{\delta_2}\right) + s_1 n_1 + s_2 n_2 \geq 0. \qquad (2.7b)
\end{aligned}
$$

For external stability condition, similarly:

$$
\begin{aligned}
w_1^{NC}(k) - w_1^C(k+1) = & \frac{1}{2\delta_1}\left(n_1\beta_1 + n_2\beta_2\right)^2 - \beta_1^2\left(\frac{n_1}{\delta_1} + \frac{n_2}{\delta_2}\right) + \\
& - s_1(n_1+1) - s_2 n_2 \geq 0; \qquad (2.8a)
\end{aligned}
$$

$$
\begin{aligned}
w_2^{NC}(k) - w_2^C(k+1) = & \frac{1}{2\delta_2}\left(n_1\beta_1 + n_2\beta_2\right)^2 - \beta_2^2\left(\frac{n_1}{\delta_1} + \frac{n_2}{\delta_2}\right) + \\
& - s_1 n_1 - s_2(n_2+1) \geq 0. \qquad (2.8b)
\end{aligned}
$$

2.3.1 Existence of a Stable Coalition

We want to find the size of a stable coalition by solving the system of inequalities (2.7a), (2.7b), (2.8a) and (2.8b) that is not analytically tractable in general. We suppose a proportionality between the parameters as follows

$$
\frac{\beta_1}{\beta_2} = \frac{\delta_2}{\delta_1} = \frac{s_1}{s_2} = \sigma
$$

for a natural numer $\sigma \in \mathcal{N}$.

Thus, if σ is greater than one, we have that $\beta_1 > \beta_2$, $\delta_2 > \delta_1$ and $s_1 > s_2$. This implies that if countries of kind 1 have more vulnerability to environmental damage, than they have a smaller marginal shift parameter and a greater inclination to join in coalition. We want highlight that the implication $\delta_1 < \delta_2$, doesn't mean that countries of kind 2 have an higher production function, because of the presence of parameters α_1 and α_2. We start by considering the condition (2.7a)

$$-\frac{1}{2\delta_1}\Big(h_1\beta_1 + h_2\beta_2 - \beta_1\Big)^2 + \beta_2^2\Big(\frac{h_1 - 1}{\delta_1} + \frac{h_2}{\delta_2}\Big) + s_1h_1 + s_2h_2 \geq 0.$$

After some rearrangements we have

$$-\frac{\beta_2^2}{2\delta_1}(h_1\sigma + h_2 - \sigma)^2 + \frac{\beta_1^2}{\delta_2}(h_1\sigma + h_2 - \sigma) + s_2(\sigma h_1 + h_2) \geq 0;$$

in which we define

$$y = h_1\sigma + h_2,$$

and than the condition become

$$-y^2 + \frac{2\delta_1}{\beta_2^2}\Big(\frac{\sigma\beta_2^2}{\delta_1} + \frac{\beta_1^2}{\delta_2} + s_2\Big)y - \sigma^2\frac{\beta_2^2}{2\delta_1}\frac{2\delta_1}{\beta_2^2} - \sigma\frac{\beta_1^2}{\delta_2}\frac{2\delta_1}{\beta_2^2} \geq 0.$$

Simplifying and rewriting in term of σ, we have

$$-y^2 + 2\Big(\sigma + \sigma + \frac{\delta_1 s_2}{\beta_2^2}\Big)y - \sigma^2 - 2\sigma^2 \geq 0,$$

and if we denote by

$$t = \frac{\delta_1 s_2}{\beta_2^2}, \qquad t > 0,$$

we have the inequality

$$-y^2 + 2(2\sigma + t)y - 3\sigma^2 \geq 0. \tag{2.9}$$

Let us call $y_1^{(2.7a)}, y_2^{(2.7a)}$ the solutions of the equation $-y^2 + 2(2\sigma + t)y - 3\sigma^2 = 0$:

$$y_1^{(2.7a)} = 2\sigma + t - \sqrt{(2\sigma + t)^2 - 3\sigma^2},\ y_2^{(2.7a)} = 2\sigma + t + \sqrt{(2\sigma + t)^2 - 3\sigma^2}$$

Similarly for condition (2.7b) we have

$$-y^2 + 2(2 + t\sigma)y - 3 \geq 0. \tag{2.10}$$

Let us call $y_1^{(2.7b)}, y_2^{(2.7b)}$ the solutions of the equation $-y^2 + 2(2 + t\sigma)y - 3 = 0$

$$y_1^{(2.7b)} = 2 + t\sigma - \sqrt{(2 + t\sigma)^2 - 3},\ y_2^{(2.7b)} = 2 + t\sigma + \sqrt{(2 + t\sigma)^2 - 3}$$

We follow the same way to reduce the external stability inequalities in a more simple form. So, for condition (2.8a) we have

$$y^2 - 2(t + \sigma)y - 2t\sigma \geq 0, \tag{2.11}$$

and, if we call $y_1^{(2.8a)}, y_2^{(2.8a)}$ the solutions of the equation $y^2 - 2(t + \sigma)y - 2t\sigma = 0$, we found

$$y_1^{(2.8a)} = t + \sigma - \sqrt{(t + \sigma)^2 + 2t\sigma},\ y_2^{(2.8a)} = t + \sigma + \sqrt{(t + \sigma)^2 + 2t\sigma}$$

For condition (2.8b) we have

$$y^2 - 2(1 + t\sigma)y - 2t\sigma \geq 0. \tag{2.12}$$

So, the solutions $y_1^{(2.8b)}, y_2^{(2.8b)}$ of the equation $y^2 - 2(1 + t\sigma)y - 2t\sigma = 0$, are

$$y_1^{(2.8b)} = 1 + t\sigma - \sqrt{(1 + t\sigma)^2 + 2t\sigma},\ y_2^{(2.8b)} = 1 + t\sigma + \sqrt{(1 + t\sigma)^2 + 2t\sigma}$$

The solutions of inequalities (2.9) and (2.10) are the intervals $[y_1^{(2.7a)}, y_2^{(2.7a)}]$, $[y_1^{(2.7b)}, y_2^{(2.7b)}]$ respectively and the solutions of inequalities (2.11) and (2.12) are the complements of the intervals $]y_1^{(2.8a)}, y_2^{(2.8a)}[$ and $]y_1^{(2.8b)}, y_2^{(2.8b)}[$. Let us note that $y_1^{(2.8a)} < 0$ and $y_1^{(2.8b)} < 0$, while $y_1^{(2.7a)}, y_2^{(2.8a)}, y_2^{(2.8b)}, y_2^{(2.7b)}, y_2^{(2.7a)} > 0$.

Proposition (Existence of a stable coalition) We have that

$$y_2^{(2.7b)} - y_2^{(2.8b)} = 1$$

Moreover, assume that $\sigma \geq 1, t \geq 1$ and if $\sigma = 1$ or $t = 1$ we have

$$y_2^{(2.8a)} \leq y_2^{(2.8b)} \leq y_2^{(2.7b)} \leq y_2^{(2.7a)}$$

and there exists a natural number $\bar{y} \in [y_2^{(2.8b)}, y_2^{(2.7b)}]$ satisfying the four inequalities (2.9), (2.10), (2.11) and (2.12).

Proof.

$$y_2^{(2.7b)} - y_2^{(2.8b)} = 2 + t\sigma + \sqrt{(2+t\sigma)^2 - 3} - [1 + t\sigma + \sqrt{(1+t\sigma)^2 + 2t\sigma}] = 1$$

because

$$(2 + t\sigma)^2 - 3 = (1 + t\sigma)^2 + 2t\sigma$$

for any σ and any t. Then $y_2^{(2.8b)} \leq y_2^{(2.7b)}$.

The inequality $y_2^{(2.8a)} \leq y_2^{(2.8b)}$, i.e.

$$t + \sigma + \sqrt{(t+\sigma)^2 + 2t\sigma} \leq 1 + t\sigma + \sqrt{(1+t\sigma)^2 + 2t\sigma}$$

is true because $\sigma + t \leq 1 + t\sigma$ for $\sigma \geq 1$ and $t \geq 1$.

The inequality $y_2^{(2.7b)} \leq y_2^{(2.7a)}$, i.e.

$$2 + t\sigma + \sqrt{(2+t\sigma)^2 - 3} \leq 2\sigma + t + \sqrt{(2\sigma+t)^2 - 3\sigma^2}$$

is true because if we assume $t = 1$ we have

$$2 + \sigma + \sqrt{(2+\sigma)^2 - 3} \leq 2\sigma + 1 + \sqrt{(2\sigma+1)^2 - 3\sigma^2}$$

then

$$2 + \sigma \leq 2\sigma + 1$$

that is true for $\sigma \geq 1$. Similarly the case $\sigma = 1$ for any $t \geq 1$ (the assumption $\sigma = 1$ gives the symmetry between developed and developing countries). For the case $\sigma \in (0, 1)$ the solutions are the same, and the proof is similar to $\sigma > 1$, we just need to invert the parameters' relations. $\square$

We have shown that, under some assumptions on the parameters, a solution exists, but we did no discuss about the size and the composition of a stable coalition. Note that there is no a unique stable coalition since all coalitions made by n_1 developed countries and n_2 developing countries such that $n_1\sigma + n_2 = \bar{y}$ are stable coalitions. In order to make more considerations, we in the following some numerical simulations.
First of all we need to calibrate the parameters, and we have to choose values which respect the constraint $e_i > 0$ for each $i \in I$. So, for $i = \{1, 2\}$, we take $\alpha_i \in [5, 10]$, $\delta_i \in [0.3, 0.9]$, $\beta_i \in [0.001, 0.01]$. Moreover, we assume $s_i \in [0.3, 0.9]$ and that we have 100 countries, equally distributed between developed and developing countries, than $n_1 \in [0, 50]$ and $n_2 \in [0, 50]$, clearly n_1 and n_2 are both natural numbers.

Example 2.1. *In this first example, we choose parameters such that $\beta_1 > \beta_2$, $\delta_2 > \delta_1$ and $s_1 > s_2$.*
So, we take

$$\delta_1 = 0.3, \quad \delta_2 = 0.5, \quad \beta_1 = 0.01, \quad \beta_2 = 0.001, \quad s_1 = 0.5, \quad s_2 = 0.3.$$

We test 64 different combinations of n_1 and n_2, assuming $n_1 = 1 + 7j$ and $n_2 = 1 + 7p$. The combinations of n_1 and n_2 are given by all the possible permutations of the indeces j and p, where $j, p = \{1, \dots, 7\}$. We summarize the results in two cartesian diagram, the first in figure 2.1 shows the internal and external stability conditions for developed countries, while in figure 2.2 we have the internal and external stability conditions for developed countries.
Figures must be read as follows: on the x axis we have the combinations of n_1 and n_2 given by the 64 permutations of the indeces j, p. So, $x = 1$ is given by $j = 0, p = 0$; $x = 2$ is given by $p = 0, j = 1$ and so on.
On the y axis there are the values of stability conditions, for each combination of n_1 and n_2. Then, the stability conditions are verified if the points lie in first quadrant, while are not verified if lie in the fourth quadrant. From figure 2.1 we can see that every coalition considered is internally stable for developed countries, but is not externally stable. Then, the result is that all the players of kind 1 outside coalition have an incentive to join. We want remark that, from [PZ13], without externality the maximal size of a stable coalition can be achieved with $n_1 = 2$.
Considering figure 2.2 we have that also for developing countries the internal stability condition is always verified, while the external one is not never. So, all the developing

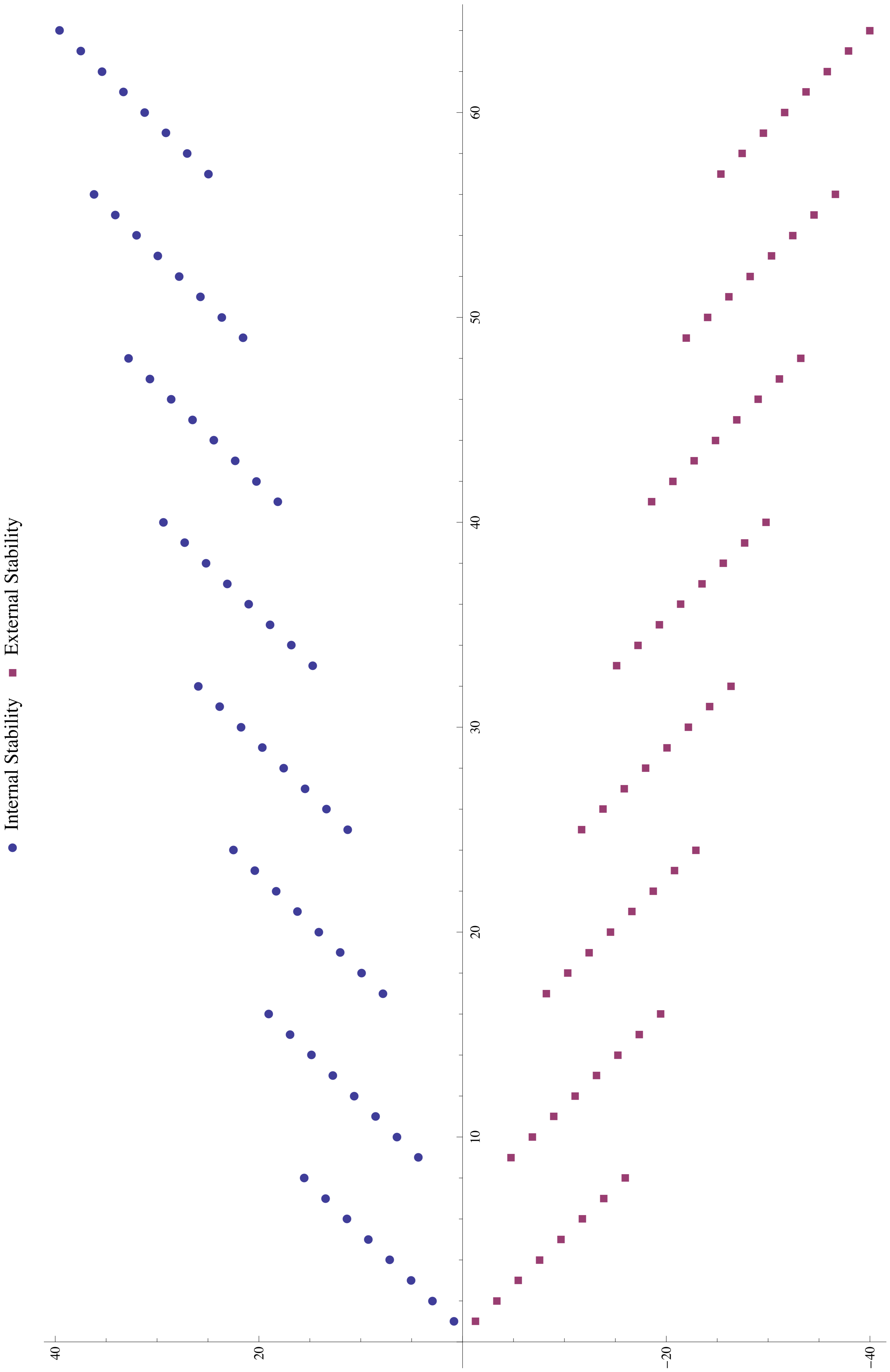

FIGURE 2.1: Example 2.1: Internal and External Stability Conditions for developed countries

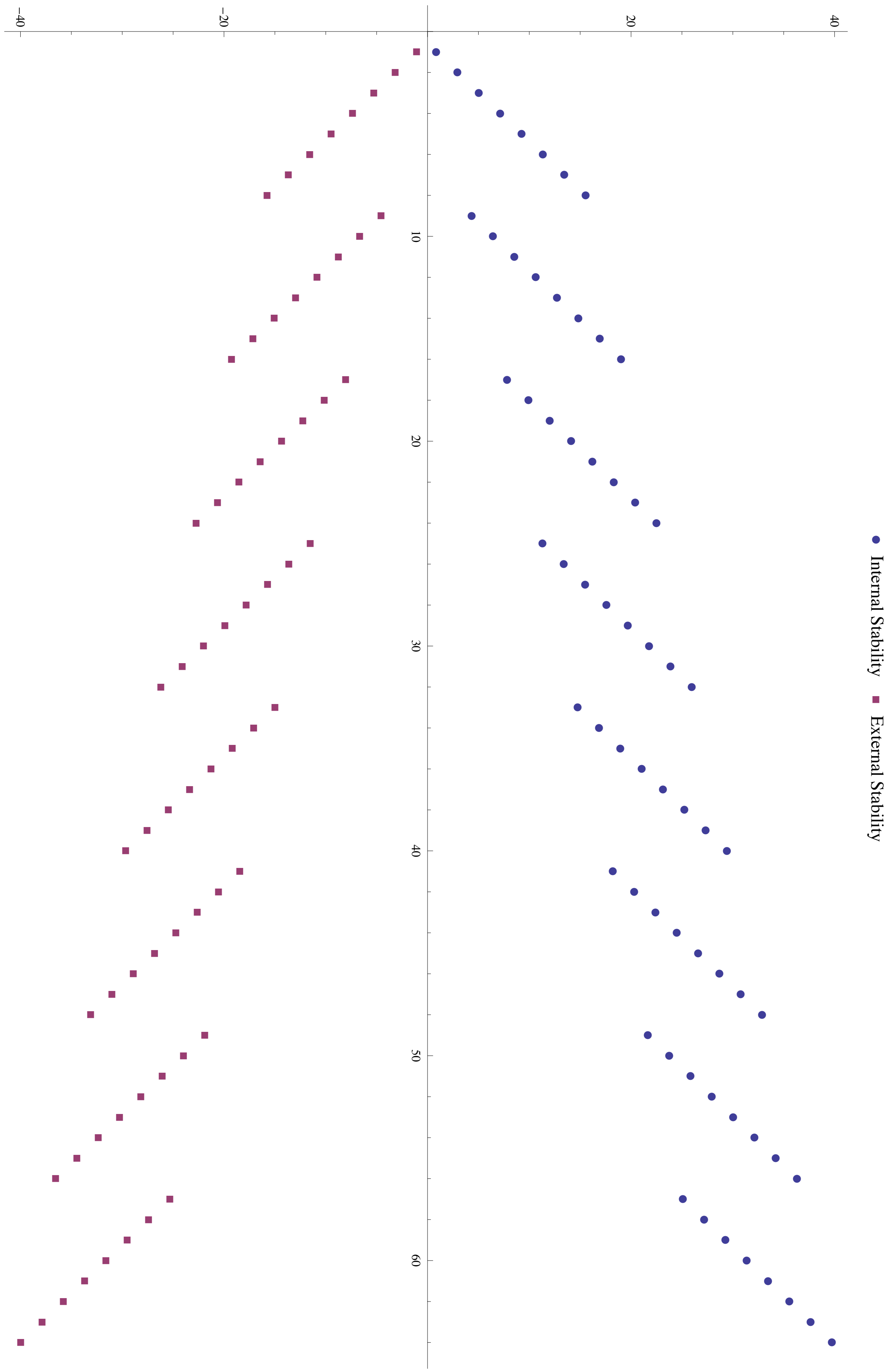

FIGURE 2.2: Example 2.1: Internal and External Stability Conditions for developing countries

countries have an incentive to join in.

Follows that we can observe a case in which the grand coalition is the only stable coalition.

Example 2.2. *In the second example we want reverse the relations between parameters, so we take*

$$\delta_1 = 0.5, \quad \delta_2 = 0.3, \quad \beta_1 = 0.001, \quad \beta_2 = 0.01, \quad s_1 = 0.3, \quad s_2 = 0.5.$$

We test the 64 combinations given by the permutation of the indeces j and p of $n_1 = 1+7j$ and $n_2 = 1 + 7p$, as in example 2.1.

We report the results of our simulations in figures 2.3 and 2.4. From figure 2.3 we can see that for developed countries every coalition is internally stable, but there is not a coalition which gives external stability.

In figure 2.4 we show the results for developing countries. The conclusion is the same: all developing countries in coalition have no incentive to defect, while all those out have an incentive to join.

Then, we have another example in which the only stable coalition is the grand coalition.

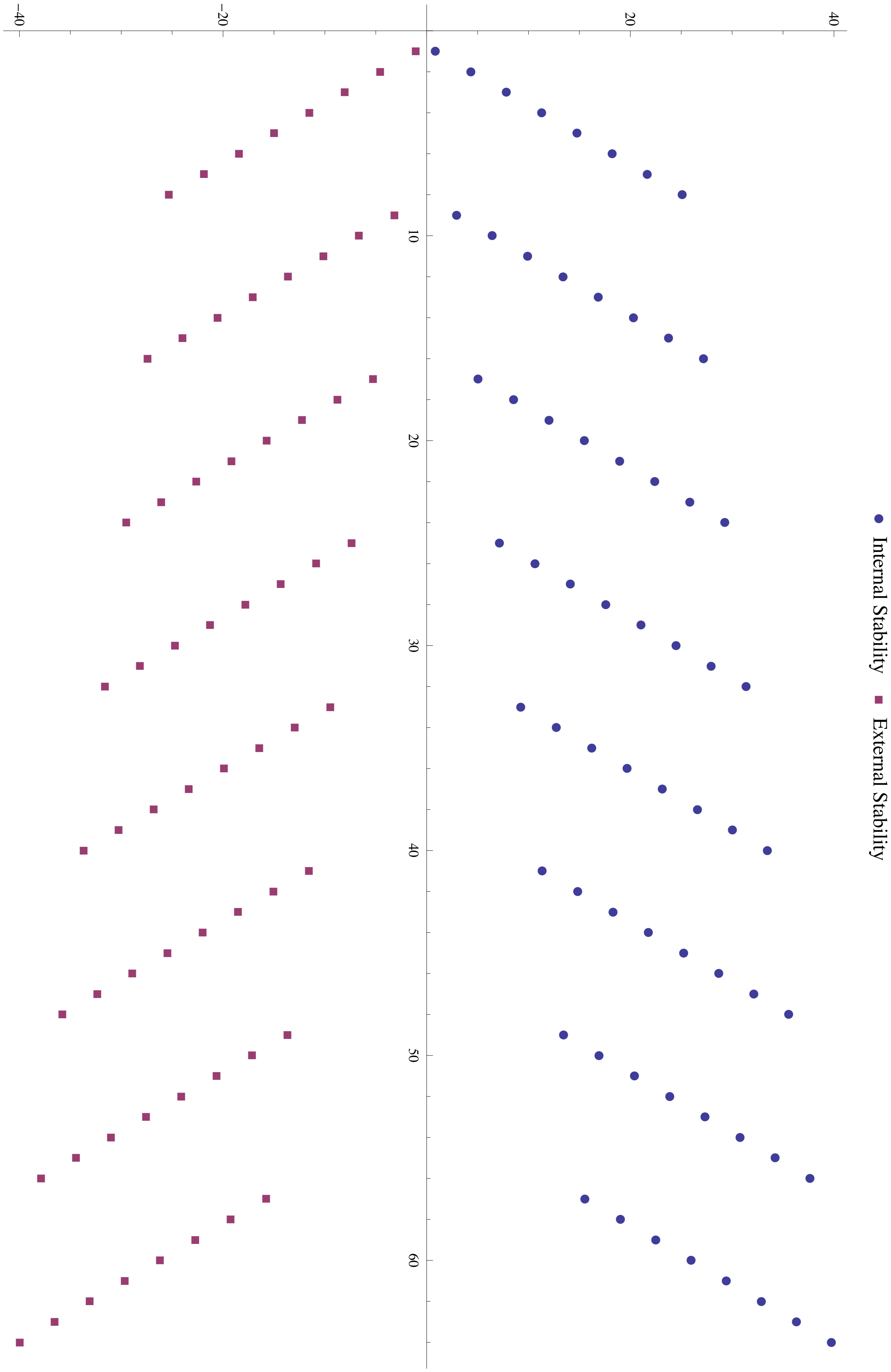

FIGURE 2.3: Example 2.2: Internal and External Stability Conditions for developed countries

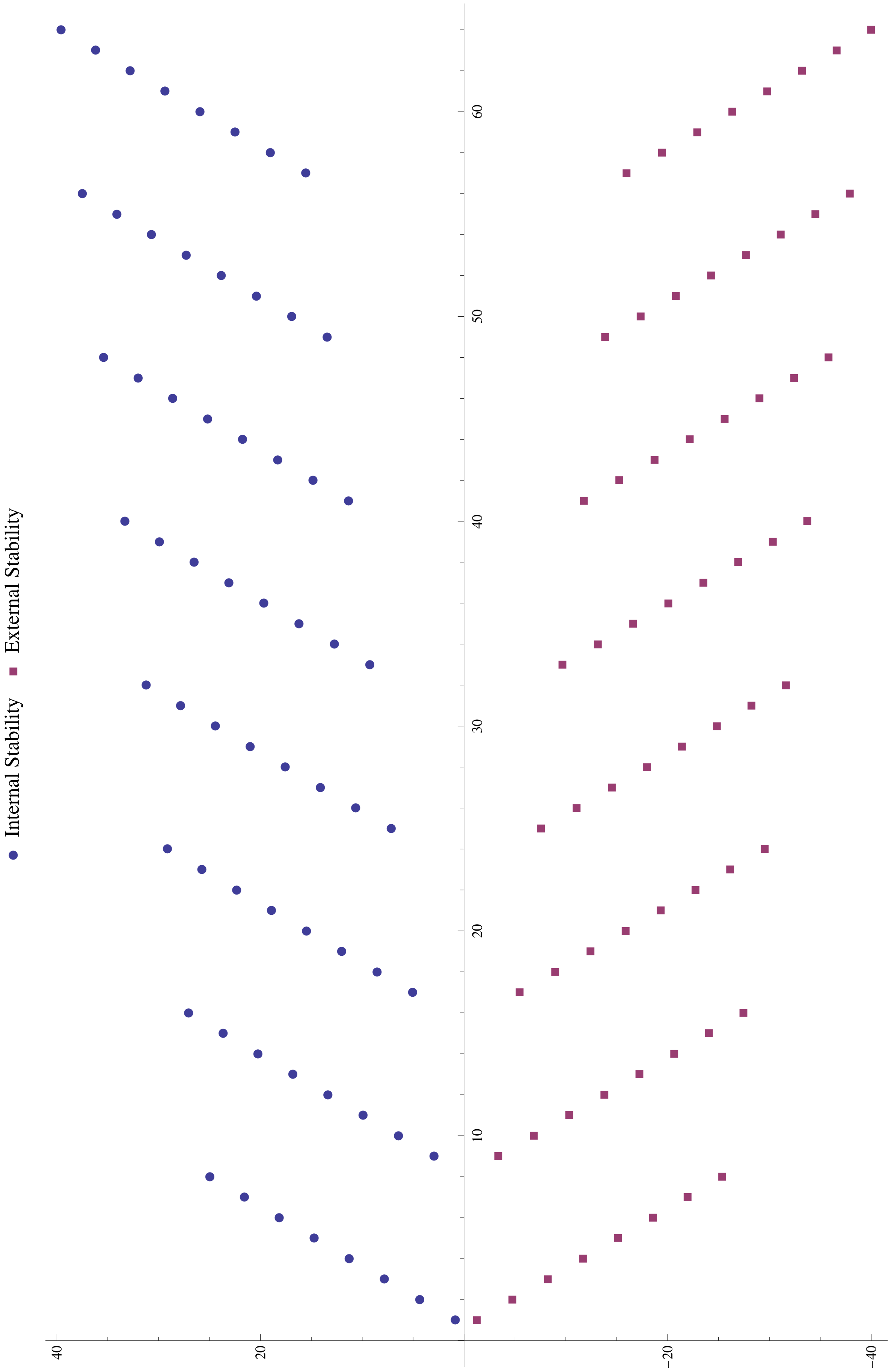

FIGURE 2.4: Example 2.2: Internal and External Stability Conditions for developing countries

To try to explain the results of the last two examples, we want highlight that the social externality acts to make stable the coalition from inside, because of has a positive effect on internal stability conditions. On the other side, it works to make unstable the coalition from outside, because of has a negative effect on external stability conditions. Than, if the externality is larger enough the result is that the players inside coalition have no incentive to defect from cooperation, while all players outside have no incentive to stay out. So, we can have that all players join in coalition.

Chapter 3

Two-player Differential Games

In this Chapter we analyze the effects of a Social Externality in a dynamical context. We consider the differential framework presented in [MZ13]. We have two players, one representative of the developed countries while the other represents developing countries. We assume that the latter player has a gradual involvement in environmental concern, which is connected with economic growth. We assume that players start acting non-cooperatively and that there is a time in the short run in which they decide to sign an agreement. In this way, we can consider also the negotiation time.

We characterize the non-cooperative and cooperative solutions of the global emission game by using Dynamic Programming method. After that, we study if the coalition of the two players is stable in the short run. Different from the static model, here the Externality function which affects the cooperative welfares, is a function of time. The Chapter is divided as follows: is Section 1 we present the model and its functional form; in Section 2 we find the emission solutions and in Section 3 we analyze the stability conditions, showing some numerical results.

3.1 The Model

We consider a two-player differential game, in which player 1 represents developed countries and player 2 represents developing countries. The main difference between two players is that for player 2 the environmental protection is not on the top of economic agenda. So, we characterize two different damage-cost functions, assuming that for player 1 the damage-cost is full from the outset, while for player 2 is an increasing function of time and become full when the player achieves a preset threshold in discounted revenues. The idea of linking environmental concern and income of a country is not new in literature (see e.g., [SB92] for an empirical analysis). Moreover, this approach is in the spirit of Kyoto protocol, in which reducing emissions is responsibility of developed countries. The motivations of this approach are both historical and economical. From the first point of view, we have that the actual level of pollution can not be attributed to developing countries, so it's not fair ask them to assume the commitment to reduce it. On the other side, developing countries have some economic priorities (e.g., improve infrastructures, increase per capita wealth and instruction level), so that environment is a luxury good.

We make the usual assumption that emissions are a by-product of production, and for a suitable production function we can use emissions like a inputs of industrial process. So, for each player i, $i = 1, 2$, we can write the production function as: $f_i(e_i)$, where e_i are the emission of player i. Then, if we denote we ρ the rate of time preference, we can define with T the instant ($T > 0$), at which player 2 become developed, as the time at which is verified:

$$\int_0^T f_i(e_i)e^{-\rho t}dt = \bar{Y_2},$$

where, $\bar{Y_2}$ is the threshold chosen.

Then, for player 2 the damage-cost function is described for any t in two intervals:

$$D_2(S(t), Y_2(t)) = \begin{cases} d_2(S(t), Y_2(t)) & \forall Y_2(t) < \bar{Y_2}, \\ \\ D_2(S(t)) & \forall Y_2(t) \geq \bar{Y_2}, \end{cases}$$

for d_2 and D_2 suitable functions.

We have that the stock of pollutant, called S, follows the dynamics

$$\begin{cases} \dot{S} = \mu(e_1 + e_2) - SS, \\ S(0) = S_0, \end{cases}$$

where μ is a positive scale parameter, δ is the positive natural absorption rate of pollution and S_0 is the given initial level of pollution stock. The assumption of rational players leads to consider that each player maximizes his discounted welfare, that we denote by w:

$$\max_{e_1} w_1 = \max_{e_1} \int_0^\infty e^{-\rho t}(f_1(e_1) - D_1(S))dt,$$
$$\max_{e_2} w_2 = \max_{e_2} \int_0^T e^{-\rho t}(f(e_2) - d_2(S, Y_2))dt + \int_T^\infty e^{-\rho(t-T)}(f_2(e_2) - D_2(S))dt,$$

subject to

$$\dot{S} = \mu(e_1 + e_2) - \delta S, \qquad S(0) = S_0.$$

3.1.1 Functional Forms and Maximization Problems

For the functional forms of the model we assume a quadratic function for production function, that's a common choice in literature (see e.g., [STZ11] and [DL93]), while for damage-cost function we take linear functions of S, as follows

$$f_i(e_i) = \alpha_i e_i - \frac{1}{2}e_i^2,$$
$$D_1(S) = \beta_1 S,$$
$$D_2(S(t), Y_2(t)) = \begin{cases} d_2(S, Y_2) = \frac{t}{T}\gamma\beta_2 S, & \forall Y_2(t) < \bar{Y_2}, \\ D_2(S) = \beta_2 S & \forall Y_2(t) \geq \bar{Y_2}, \end{cases}$$

where $e_i \in \Delta$, with Δ that is the strategy space (Δ is a real interval), and the parameters α_i are strictly positive. So, we implicitly make the assumption that $f_i(e_i)$ is strictly positive for every possible values of e_i. In the damage-cost function we assume that β_i are strictly positive parameters. The assumption of linearity is a simplification, but we expect that its effects are quantitative and not qualitative. Moreover, we assume $\gamma \in \{0, 1\}$. The case $\gamma = 0$ is when player 2 have not at all consideration about environmental issue, while $\gamma = 1$ is the case of gradual involvement. Finally, we assume that the externality function is a quadratic and decreasing function of time

$$Ext(t) = -t^2 + t^\star,$$

where $t^\star \in [0, T]$ is the starting time of cooperation. The hypothesis is that when countries have make the decision to join or not an agreement, they consider all possible earnings due to relations with other countries. Than, we add the positive function $Ext(t)$ to the cooperative welfare. We want remark that the externality is not related with the control variable e_i, so don't affects the maximization process.
The assumption of a decreasing externality deals with the will of non overestimate its effect on the stability of the coalition.

We assume that exists a time $t^\star \in [0, T]$ at which the two players begin to cooperate. We can interpret the interval $[0, t^\star]$ as the period in which player 1 and player 2 negotiate the environmental agreements. We have the players maximize non-cooperatively in $[0, t^\star]$ and maximize the joint welfare in $[t^\star, \infty)$, as follows

- for $t \in [0, t^\star]$:

$$\max_{e_1} \int_0^{t^\star} (f_1(e_1) - D_1(S))e^{-\rho t} dt,$$

$$\max_{e_2} \int_0^{t^\star} (f_2(e_2) - d_2(S, Y_2))e^{-\rho t} dt,$$

- for $t \in [t^\star, \infty)$:

$$\max_{e_1, e_2} \left\{ \int_{t^\star}^{T} (f_1(e_1) + f_2(e_2) - D_1(S) - d_2(S, Y_2) + Ext(t))e^{-\rho(t-t^\star)} dt + \right.$$
$$\left. + \int_T^{\infty} (f_1(e_1) + f_2(e_2) - D_1(S) - D_2(S) + Ext(t))e^{-\rho(t-T)} dt \right\},$$

both problems subject to

$$\begin{cases} \dot{S} = \mu(e_1 + e_2) - \delta S, \\ S(0) = S_0. \end{cases}$$

3.2 Emission Solutions

In order to find the feedback Nash equilibrium we proceed to solve the optimization problems backward. So, we first characterize the cooperative solutions.

3.2.1 Cooperative Solutions

Let us call $T^\star$ the instant of time at which player 2 becomes developed with optimal emission.

We consider first the case of the interval $[T^\star, \infty)$. The optimization problem is given by

$$\begin{aligned} &\max_{e_1,e_2} \int_{T^\star}^{\infty} (f_1(e_1) + f_2(e_2) - D_1(S) - D_2(S) + Ext(t))e^{-\rho t} dt, \\ &\text{subject to: } \dot{S}(t) = \mu(e_1 + e_2) - \delta S, \quad S(0) = S_0. \end{aligned} \tag{3.1}$$

Let $v(t, S)$ the value function, we have the Hamilton-Jacobi-Bellman (HJB) equation

$$-v_t = \max_{e_1,e_2} \left\{ \left(\alpha_1 e_1 + \alpha_2 e_2 - \frac{1}{2}(e_1^2 + e_2^2) - S(\beta_1 + \beta_2) + Ext(t) \right) e^{-\rho t} + v_s(\mu(e_1 + e_2) - \delta S) \right\}. \tag{3.2}$$

Assuming an interior solution, the maximization of the right-hand of the equation gives

$$e_i = \alpha_i + \mu v_s e^{\rho t}.$$

We assume that value function is linear in S, as follows

$$v(t, S) = (KS + L(t))e^{-\rho t},$$

which has the following partial derivatives: $v_t = -\rho(KS + L(t) - L'(t))e^{-\rho t}$ and $v_S = Ke^{-\rho t}$.

So, the emission solution is given by $e_i = \alpha_i + K\mu$.

Substituting the expression of e_i, v_s and v_t in (3.1), for the principle of identity of polynomials we have

$$\rho K = -\beta_1 - \beta_2 - \delta K,$$

$$K = -\frac{\beta_1 + \beta_2}{\rho + \delta}.$$

Then, for each player i, in the interval $[T,\infty)$, the optimal emissions $e_i^\star$ are

$$e_i^\star(t) = \alpha_i - \mu\frac{\beta_1+\beta_2}{\rho+\delta}, \qquad i=1,2. \tag{3.3}$$

Consider now the interval $[t^\star, T^\star]$.

Due to the change in damage cost function of player 2, the optimization problem is

$$\max_{e_1,e_2}\int_{t^\star}^{T^\star}\left[(\alpha_1 e_1+\alpha_2 e_2-\frac{1}{2}(e_1^2+e_2^2)-S\left(\beta_1+\gamma\beta_2\frac{t}{T^\star}\right)+Ext(t)\right]e^{-\rho t}dt,$$

subject to $\dot{S}(t)=\mu(e_1+e_2)-\delta S, \quad S(0)=S_0$.

Called $v(t,S)$ the value function, we have the Hamilton-Jacobi-Bellman equation

$$-v_t=\max_{e_1,e_2}\left\{\left(\alpha_1e_1+\alpha_2e_2-\frac{1}{2}(e_1^2+e_2^2)-S\left(\beta_1+\gamma\beta_2\frac{t}{T^\star}\right)+Ext(t)\right)e^{-\rho t}+v_s(\mu(e_1+e_2)-\delta S)\right\}. \tag{3.4}$$

From the first order conditions in (3.4) we can derive the expression

$$e_i(t)=\alpha_i+\mu v_S e^{\rho t}.$$

We assume a value function linear in S, with non-constant coefficients, as follows

$$v(t,S)=[g(t)S+z(t)]e^{-\rho t},$$

whose partial derivatives respect t and S are

$$v_t=[(g'(t)-\rho g(t))S+z'(t)-z(t)]e^{-\rho t}, \qquad v_S=g(t)e^{-\rho t}.$$

The expression of v_S give us the emission solutions

$$e_i(t)=\alpha_i+\mu g(t), \qquad i=1,2.$$

To find the solution we need to substitute emission and derivatives of value function in (3.4). After some algebra, we derive the differential equation

$$\begin{cases}\rho g(t)-g'(t)=-\beta_1-\gamma\frac{t}{T^\star}\beta_2-\delta g(t)\\ g(T^\star)=K=-\frac{\beta_1+\beta_2}{\rho+\delta},\end{cases}$$

which has the unique solution

$$g(t) = -\frac{\beta_1}{\rho+\delta} - \frac{\beta_2}{T^\star(\rho+\delta)^2}\big[\gamma\big(1+t(\rho+\delta)-e^{(\rho+\delta)(t-T^\star)}\big) - e^{(\rho+\delta)(t-T^\star)}T^\star(\rho+\delta)(\gamma-1)\big].$$

So, optimal emission for player i in the interval $[t^\star, T^\star]$ are given by

$$e_i^\star(t) = \alpha_i - \mu\frac{\beta_1}{\rho+\delta} - \mu\frac{\beta_2}{T^\star(\rho+\delta)^2}\big[\gamma\big(1+t(\rho+\delta)-e^{(\rho+\delta)(t-T^\star)}\big) - e^{(\rho+\delta)(t-T^\star)}T^\star(\rho+\delta)(\gamma-1)\big], \tag{3.5}$$

where $i = 1, 2$.

3.2.2 Non-Cooperative Solutions

We now consider the non-cooperative problems in the interval $[0, t^\star]$. In this interval the players maximize their own welfare, as follows

$$\max_{e_1} w_1 = \max_{e_1} \int_0^{t^\star} \Big(\alpha_1 e_1 - \frac{1}{2}e_1^2 - \beta_1 S\Big)e^{-\rho t}dt,$$

$$\max_{e_2} w_2 = \max_{e_2} \int_0^{t^\star} \Big(\alpha_2 e_2 - \frac{1}{2}e_2^2 - \gamma\beta_2\frac{t}{T^\star}S\Big)e^{-\rho t}dt.$$

In this case we have two value functions, $v^1(t,S)$ and $v^2(t,S)$, which give us the two HJB equations

$$\begin{aligned} -v_t^1 &= \max_{e_1}\Big\{\Big(\alpha_1 - \frac{1}{2}e_1^2 - \beta_1 S\Big)e^{-\rho t} + v_s^1(\mu(e_1+e_2) - \delta S)\Big\}, \\ -v_t^2 &= \max_{e_2}\Big\{\Big(\alpha_2 - \frac{1}{2}e_2^2 - \gamma\frac{t}{T^\star}\beta_2 S\Big)e^{-\rho t} + v_s^2(\mu(e_1+e_2) - \delta S)\Big\}. \end{aligned} \tag{3.6}$$

The maximization on the right-hand of the equations (3.6) leads to emissions

$$e_i(t) = \alpha_i + \mu v_S^i.$$

We assume for each player i a value function linear in S, as follows

$$v^i(t,S) = (x_i(t)S + y_i(t))e^{-\rho t},$$

which have first derivatives as follows

$$v_t = [(x_i'(t) - \rho x_i(t))S + y'(t) - y(t)]e^{-\rho t}, \qquad v_S = x_i(t)e^{-\rho t}.$$

Then, the expression of the emissions becomes

$$e_i(t) = \alpha_i + \mu x_i(t).$$

Substituting e_i, v_t and v_S inside (3.6) leads to the differential equations

$$\begin{cases} \dot{x}_1(t) = \beta_1 + (\rho+\delta)x_1(t), \\ \dot{x}_2(t) = \gamma \frac{t}{T^\star}\beta_2 + (\rho+\delta)x_2(t), \end{cases}$$

which have both the same final condition

$$x_i(t^\star) = -\frac{\beta_1}{\rho+\delta} - \frac{\beta_2}{T^\star(\rho+\delta)^2}\left[\gamma(1+t'(\rho+\delta) - e^{(\rho+\delta)(t'-T^\star)}) - e^{(\rho+\delta)(t'-T^\star)}T(\rho+\delta)(\gamma-1)\right].$$

The system has a unique solution $(x_1(t), x_2(t))$, that characterize the unique emission solution $(e_1^\star, e_2^\star)$ in the interval $[0, t^\star]$:

$$\begin{cases} x_1(t) = -\frac{\beta_1}{\rho+\delta} + \\ -\frac{\beta_2}{T^\star(\rho+\delta)^2} e^{(\rho+\delta)(t-t^\star)}\left[\gamma(1+t^\star(\rho+\delta) - e^{(\rho+\delta)(t^\star-T^\star)}) + e^{(\rho+\delta)(t^\star-T^\star)}T^\star(\rho+\delta)(1-\gamma)\right], \\ \\ x_2(t) = -\frac{\beta_1}{\rho+\delta}e^{(\rho+\delta)(t-t^\star)} + \\ -\frac{\beta_2}{T^\star(\rho+\delta)^2}\left[\gamma(1+t(\rho+\delta) - e^{(\rho+\delta)}) + e^{(\rho+\delta)(t-T^\star)}T^\star(\rho+\delta)(1-\gamma)\right]. \end{cases}$$

So, the emission solutions are given by

$$\begin{aligned}
e_1^\star(t) &= \alpha_1 - \mu\frac{\beta_1}{\rho+\delta} + \\
&- \mu\frac{\beta_2}{T^\star(\rho+\delta)^2} e^{(\rho+\delta)(t-t^\star)}\left[\gamma(1+t^\star(\rho+\delta) - e^{(\rho+\delta)(t^\star-T^\star)}) + e^{(\rho+\delta)(t^\star-T^\star)}T^\star(\rho+\delta)(1-\gamma)\right],
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
e_2^\star(t) &= \alpha_2 - \mu\frac{\beta_1}{\rho+\delta} e^{(\rho+\delta)(t-t^\star)} + \\
&- \mu\frac{\beta_2}{T^\star(\rho+\delta)^2}\left[\gamma(1+t(\rho+\delta) - e^{(\rho+\delta))} + e^{(\rho+\delta)(t-T^\star)}T^\star(\rho+\delta)(1-\gamma)\right].
\end{aligned}$$

3.3 Stability Conditions

In this section we analyze the effects of the social externality on the stability of the coalition in the short run.

We refer to the notion of internal and external stability, like in [dAs+83]. So, in the general case, a coalition formed by k players is stable if are verified the two conditions

$$w_i^C(k) \geq w_i^{NC}(k-1), \qquad w_j^{NC}(k) \geq w_j^C(k+1),$$

The first inequality, called internal stability, says that no player inside coalition has an incentive to defect, while the second inequality, called external stability, says that no player outside coalition has an incentive to join in. We have a two-player game, than we deal just with the internal stability condition. So, we have to analyze for each player if the cooperative welfare is greater than the non-cooperative welfare. We need an expression for full time non-cooperative emissions, denoted by e_1^N and e_2^N, that we take from [MZ13]

$$e_1^N(t) = \alpha_1 - \mu\frac{\beta_1}{\rho+\delta},$$
$$e_2^N(t) = \alpha_2 - \mu\frac{\beta_2}{T^N(\rho+\delta)^2}\left[\gamma(t(\rho+\delta) - e^{(\rho+\delta)(t-T^N)}) + e^{(\rho+\delta)(t-T^N)}T^N(\rho+\delta)(1-\gamma)\right],$$

where T^N is the instant of time at which player 2 becomes developed when emission is $e_2^N(t)$.

Than, the stability condition of our model are given by

$$\int_{t'}^{T^m}[f_1(e_1^\star) - D_1(S^\star) + Ext(t)]e^{-\rho(t-t^\star)}dt \geq \int_{t^\star}^{T^m} e^{-\rho(t-t^\star)}[f_1(e_1^N) - D_1(S^N)]dt,$$

$$\int_{t^\star}^{T^m}[f_2(e_2^\star) - d_2(S^\star) + Ext(t)]e^{-\rho(t-t^\star)}dt \geq \int_{t^\star}^{T^m} e^{-\rho(t-t^\star)}[f(e_2^N) - d_2(S^N)]dt,$$

where $T^m = \min\{T^\star, T^N\}$ and S^N and $S^\star$ are the pollutant stock in the non-cooperative and cooperative cases. Unfortunately, we are not able to solve the stability conditions analytically, due to the number of parameters and the high degree of non-linearity.

Then, we present some numerical results, analyzing separately the case $\gamma = 0$ and $\gamma = 1$.

If $\gamma = 0$ stability conditions are

$$\int_{t^\star}^{T^m} [f_1(e_1^\star) - \beta_1 S^\star + Ext(t)]e^{-\rho(t-t^\star)}dt \geq \int_{t^\star}^{T^m} e^{-\rho(t-t^\star)}[f_1(e_1^N) - \beta_1 S^N]dt,$$

$$\int_{t^\star}^{T^m} [f_2(e_2^\star) + Ext(t)]e^{-\rho(t-t^\star)}dt \geq \int_{t^\star}^{T^m} e^{-\rho(t-t^\star)}[f(e_2^N)]dt.$$

In the case of $\gamma = 1$ we have

$$\int_{t^\star}^{T^m} [f_1(e_1^\star) - \beta_1 S^\star + Ext(t)]e^{-\rho(t-t^\star)}dt \geq \int_{t^\star}^{T^m} e^{-\rho(t-t^\star)}[f_1(e_1^N) - \beta_1 S^N]dt,$$

$$\int_{t^\star}^{T^m} [f_2(e_2^\star) - \frac{t}{T^m}\beta_2 S^\star + Ext(t)]e^{-\rho(t-t^\star)}dt \geq \int_{t^\star}^{T^m} e^{-\rho(t-t^\star)}[f(e_2^N) - \frac{t}{T^m}\beta_2 S^N]dt.$$

In order to make some simulations, we calibrate parameters as follows: based on [Nor93], we fix $\delta = 0.0083$ and $\mu = 0.64$, with $\delta \in [0.001, 0.01]$ e $\mu \in [0.02, 0.2]$, while for the other parameters we assume $\rho \in [0.02, 0.2]$, $\alpha_i \in [0.5, 1]$ e $\beta_i \in [0.001, 0.01]$, for $i = 1, 2$. We consider 243 different combinations of parameters α_i, β_i, δ and ρ. Before proceed to analyze the stability conditions we determine the value of $T^\star$ e T^N, for all the combinations of parameters selected. Moreover, we consider different values of starting time of cooperation $t^\star \in [0, T]$.

When $\gamma = 1$ there is a large region of parameters for which coalition is stable also without externality. For example, if ρ is closer to 0.1, then the coalition is stable for all values of the other parameters.

When $\gamma = 0$ the externality have a determinant role. In fact, if $t^\star > 0$, from simulations we have $T^\star < T^N$ that implies $e_2^N > e_2^\star$, condition for which we can't have stability for player 2 without externality. Nevertheless, we have the same results also for $t^\star = 0$.

In figure 3.1, 3.2, 3.3 and 3.4 we study the case of $\gamma = 0$, with $t^\star = \frac{T^\star}{2}$. On the x axis we have all the combinations of parameters, while on the y axis we have the values of stability conditions. We consider those values as the difference between cooperative and non-cooperative welfare of each player, for each combination of parameters. Then, the stability is given when the values are positive.

When there is no externality, we observe that player 2 has no incentive to make an agreement with player 1, then the coalition is not stable. From figure 3.3 and 3.4 we have that in the same condition the presence of a social externality can enforce the agreement between player 1 and player 2.
We note that from our simulations, there is not significant difference changing the starting time $t^\star$.

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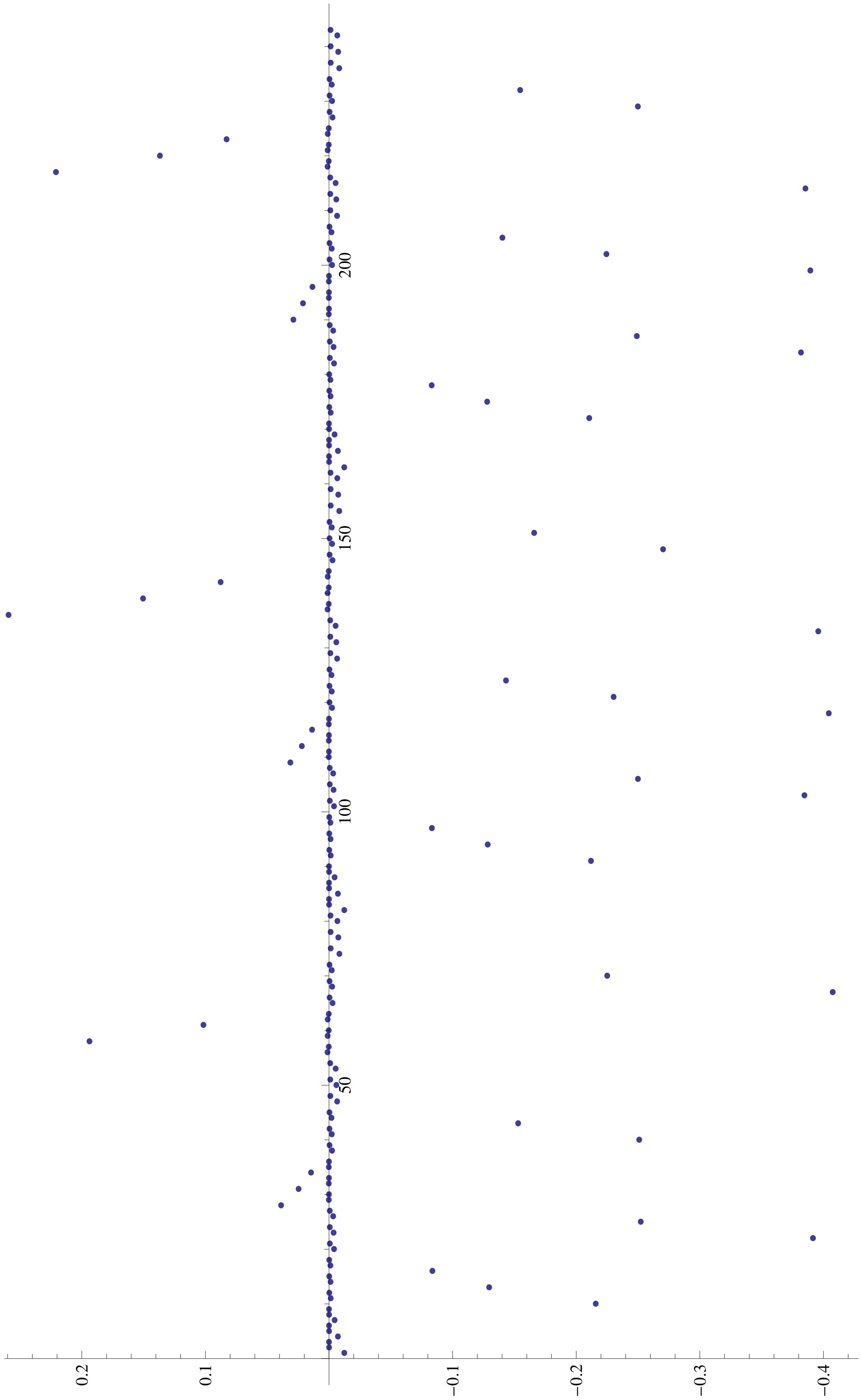

FIGURE 3.1: Stability for player 1 without externality

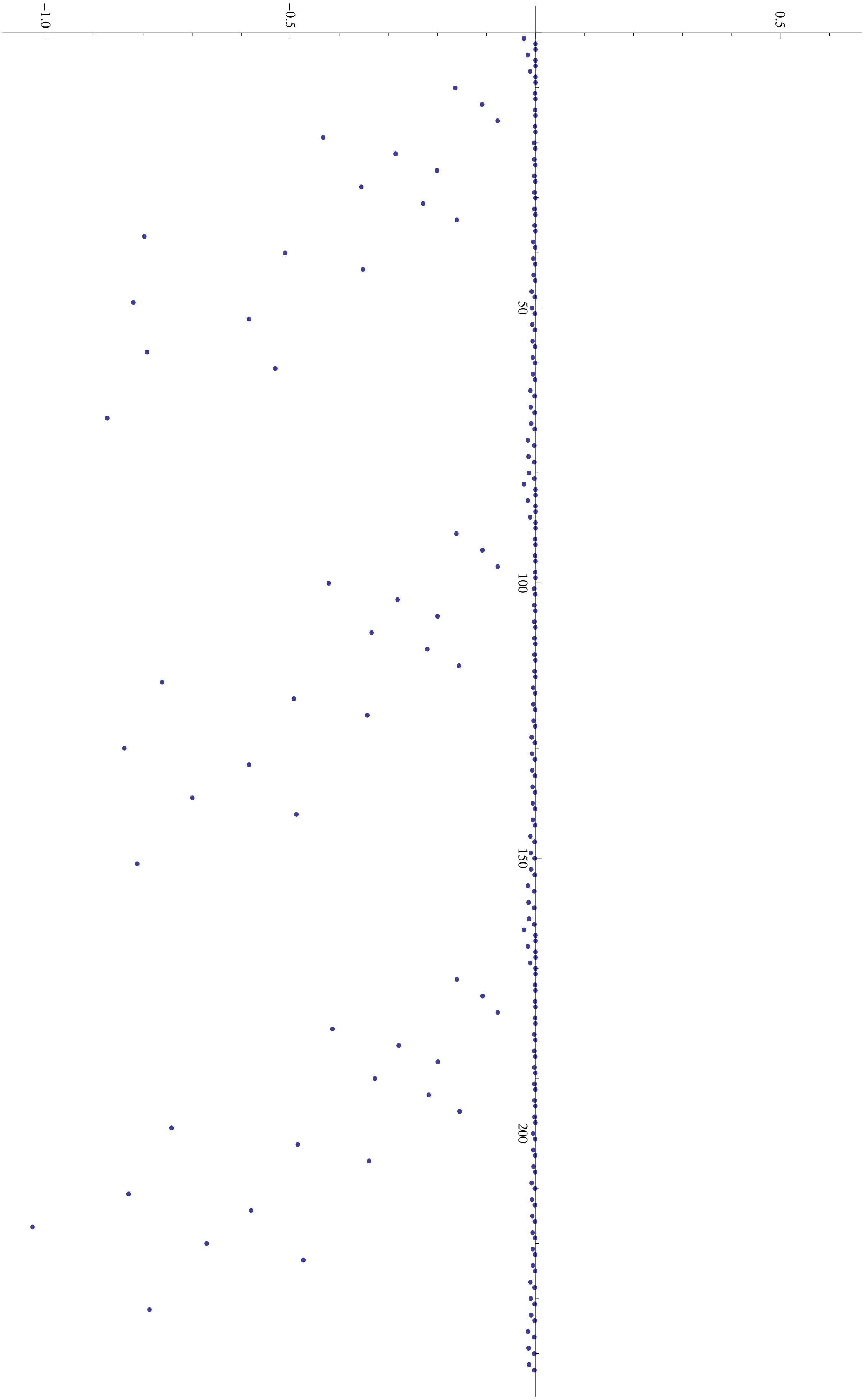

FIGURE 3.2: Stability for player 2 without externality

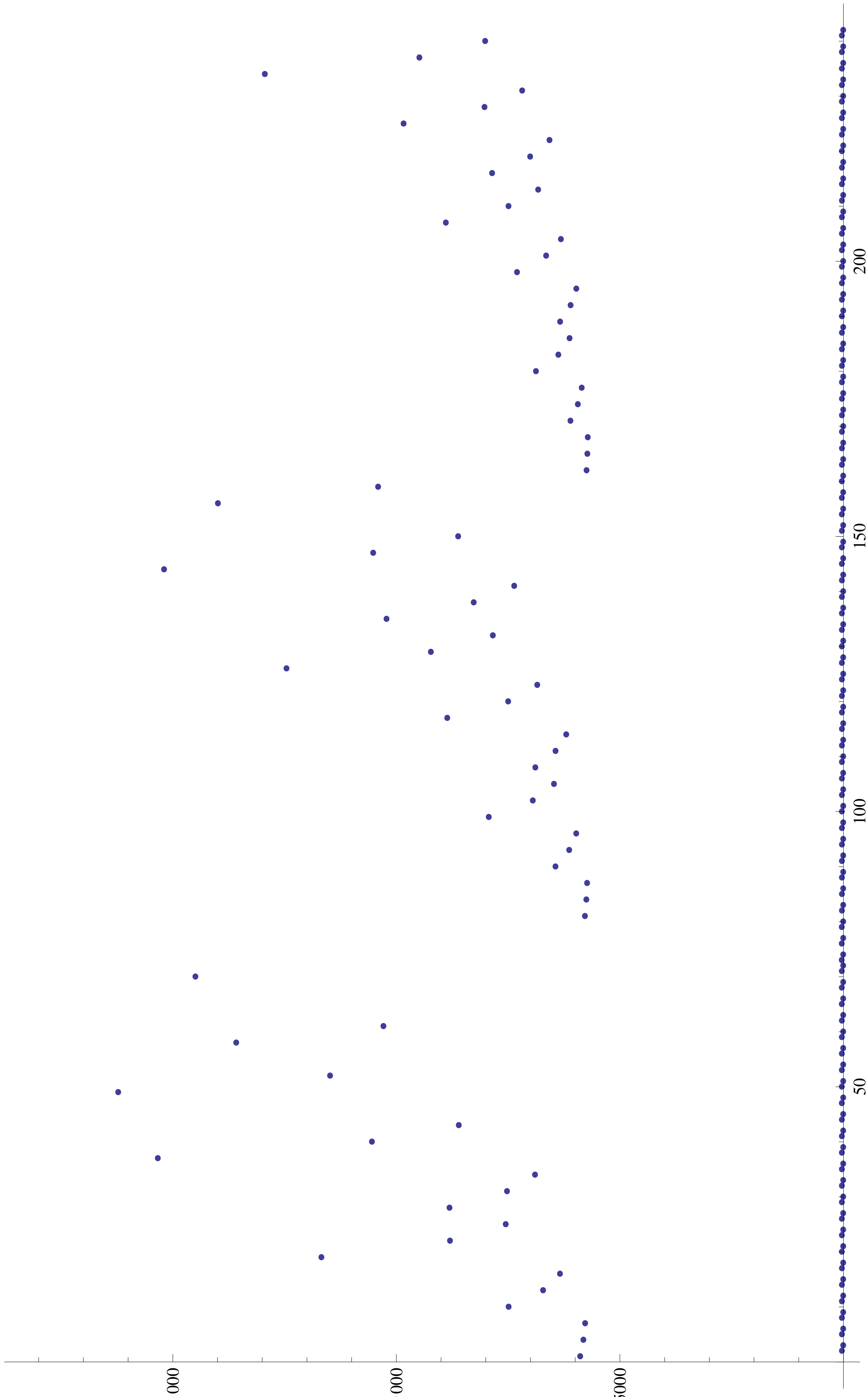

FIGURE 3.3: Stability for player 1 with externality

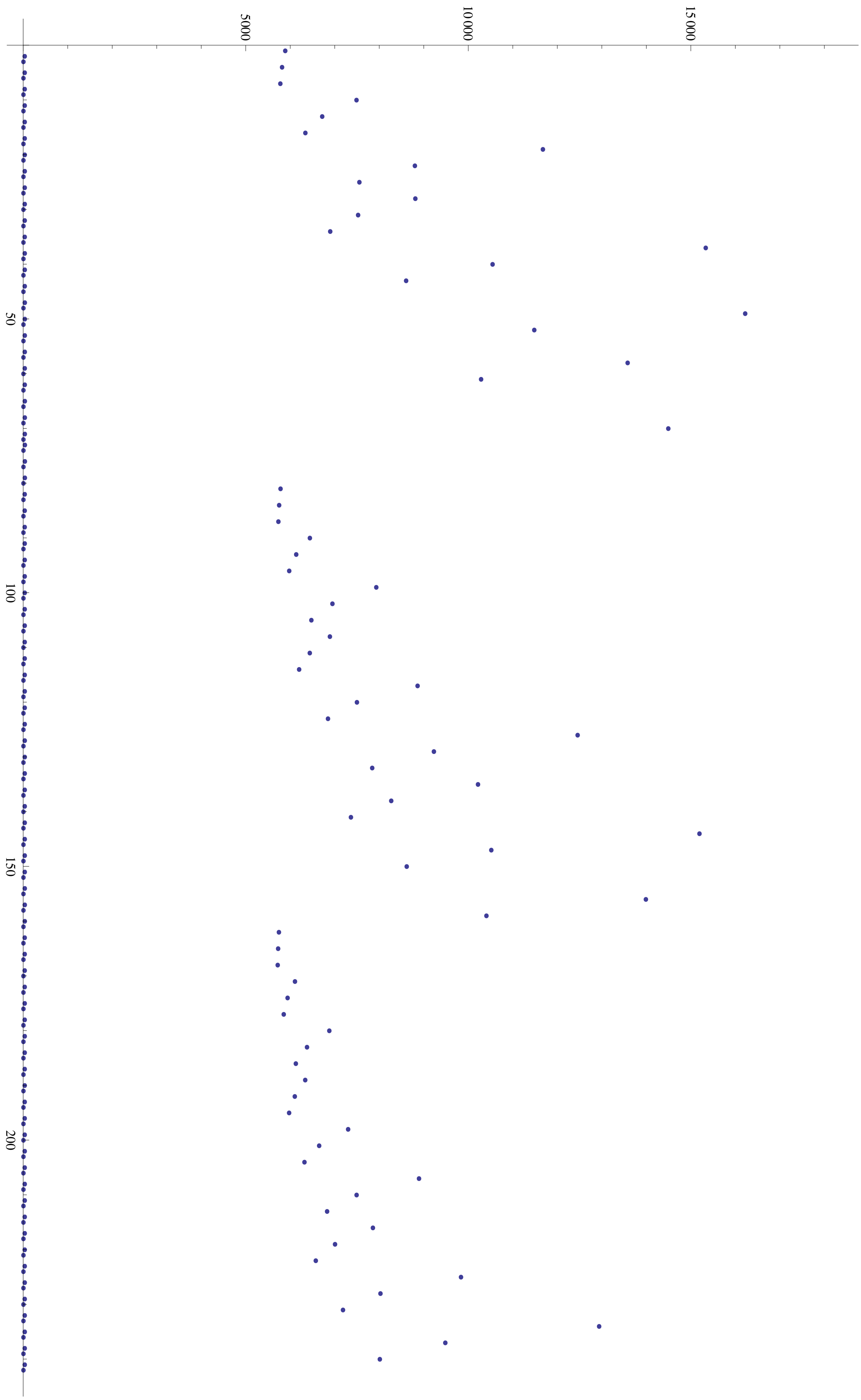

FIGURE 3.4: Stability for player 2 with externality

Chapter 4

N-player Differential Games

In this last Chapter, we will extend the two-player framework of the previous model, assuming now that we have N players. Players are divided in two different kinds, developed countries and developing countries. Within each group players are homogeneous. Respect the two-player differential game, we have an Externality function which depends on number of countries in coalition and miantain the same functional forms about production and damage-cost functions. We assume that cooperation starts from the outset of the game. As for the static model, we start by solving the global emission game, assuming a partial cooperative framework. We characterize the emission solutions both for cooperators and for defectors differentiating between developed and developing countries.

After that we discuss about the stability of the coalition, assuming that we are looking for a self-enforcing agreement. We use some numerical simulations to test if the Social Externality can lead to a stable and grand coalition. The rest of the Chapter is divided as follows: in Section 1 we present the model and its functional forms; in Section 2 we characterize the emission solution and in Section 3 we discuss about stability, showing some numerical simulations.

4.1 The Model

We consider a N-player differential game, assuming the world divided in two kinds of countries. So, we have a total number of players $N = N_1+N_2$, in which N_1 are developed countries and N_2 are developing countries (N, N_1 and N_2 are integer numbers). We use 1 to denote developed countries and 2 to denote developing countries. The hypothesis of homogeneity within the two groups, allow us to consider the game like a 4-player differential game, in which two players cooperate and two players defect. We approach this game with the notion of partial cooperative equilibrium, as we did in previous Chapters. We extend the global emission differential game investigated in [MZ13].
First of all, due to the fact that emissions are by-product of industrial activities, and assuming that the function that relates emissions and production are smooth and invertible, we can express the production for country i as a function of emission levels, and we indicate it with $f_i(e_i)$, being e_i the emission of country i.
Moreover, our hypothesis is that developed countries have an higher degree of interest in environmental issues respect developing countries, both for economic and historical motivations. In the end, this is the same approach of the Kyoto protocol.
The point is: a developing country need to improve his infrastructures, increasing per capita wealth, life expectance, instruction level, etc. In this context, environment is a "luxury good". In addition, the actual level of stock pollutant cannot be attributed to developing countries.
The power of this framework lies in the different degree of internalization of the environmental damage-cost. While the cost function, denoted by $D_i(S)$, where $S(t)$ is the stock of pollutant at time t, is full from the outset for developed countries, we suppose that for developing countries we have a gradual internalization of the cost. For this kind of players the full damage-cost is related to the achievement of a preset threshold in terms of cumulatives revenues. The idea of linking income and environmental quality is not new in literature, there's the well-known environmental Kutznets curve (EKC) and some works, like [SB92], that support the empirical consistence of this hypothesis.
So, denoting by ρ the rate of time preference, we define the time T as the instant ($T > 0$) at which a country of kind 2 becomes developed, that is the time at which is verified:

$$\int_0^T f_i(e_i)e^{-\rho t}dt = \bar{Y_2},$$

where, $\bar{Y_2}$ is the threshold chosen.

Then, for players 2 the damage-cost function is described for any t in two intervals:

$$D_2(S(t), Y_2(t)) = \begin{cases} d_2(S(t), Y_2(t)) & \forall Y_2(t) < \bar{Y_2}, \\ \\ D_2(S(t)) & \forall Y_2(t) \geq \bar{Y_2}, \end{cases}$$

for d_2 and D_2 suitable functions. The stock of pollutant $S(t)$ is solution of the following differential equation:

$$\dot{S}(t) = \mu\left(\sum_1^N e_i(t)\right) - \delta S(t), \quad S(0) = S_0,$$

where μ is a positive scaling parameter and δ is the natural rate of absorption of pollution. Here S_0 is the initial value of the pollution.

As for players 2, we denote by $D_1(S)$ the damage-cost function for developed countries. In this case, we have the same form for all the interval $[0, \infty)$. We can now introduce the payoff of player i, $i = 1, 2$, as follows:

$$w_1 = \int_0^\infty (f_1(e_1) - D_1(S))e^{-\rho t}dt,$$

and

$$w_2 = \int_0^T (f_2(e_2) - d_2(S))e^{-\rho t}dt + \int_T^\infty (f_2(e_2) - D_2(S))e^{-\rho(t-T)}dt.$$

4.1.1 Functional Forms

In this section we specify the functional forms of the model.

For production function we assume that $f_i(e_i)$ is a concave and increasing function, that's a standard assumption in literature (see, e.g., [RC05]; [DS06], [Fin01]). We choose a linear damage-cost function of stock of pollutant, that is a common choice (see [HS97], [BSZ10]) and is supported by some empirical estimation (see [LL03]). Also for externality function, denoted by $Ext(n_1, n_2)$, we make the assumption that is a positive linear function of coalition dimension (see [CR06]).

So, we have the following functional forms:

$$
\begin{aligned}
f_i(e_i) &= \alpha_i e_i - \frac{1}{2}e_i^2, \\
Ext(n_1, n_2) &= s_1 n_1 + s_2 n_2, \\
D_1(S) &= \beta_1 S, \\
D_2(S(t), Y_2(t)) &= \begin{cases} \frac{t}{T}\gamma\beta_2 S, & \forall Y_2(t) < \bar{Y_2}, \\ \\ \beta_2 S & \forall Y_2(t) \geq \bar{Y_2}, \end{cases}
\end{aligned}
$$

with $e_i \in \Delta$, where Δ is the strategy space. We impose some limits to parameters choice. Specifically, we assume α_i and β_i to be strictly positive for $i = 1, 2$, in this way we have that $f_i(e_i)$ is positive for all possible value of emissions and that the damage-cost is an increasing function of stock of pollutant. The same assumption, for the same reason, is made for the marginal externality s_i. With n_i we state the number of players of kind i that join in coalition, precisely $n_i \in \{0, 1, ..., N_i\}$, $i = 1, 2$. Moreover, we suppose $\gamma \in \{0, 1\}$. The case $\gamma = 0$ is equivalent to say that players of kind 2 don't allow for pollution at all, until they reach the threshold $\bar{Y_2}$ (that's the spirit of Kyoto protocol). If $\gamma = 1$, we are in the case of gradual internalization of damage cost.

4.2 Emission Solutions

In this section we characterize the emission solutions both for cooperators and non-cooperators. We suppose that we have a set C of cooperator players, with n_1 developed countries and n_2 developing countries. The set of non-cooperators is denoted by NC and is arranged by $NC_1 = (N_1 - n_1)$ developed countries and $NC_2 = (N_2 - n_2)$ developing countries. As usual, every player $j \in NC$ maximize his own welfare, while players in C maximize the joint welfare. Due for homogeneity within groups, we just have to find four emission solutions: two for cooperators (called e_1^C and e_2^C), and two for the defectors (called e_1^{NC} and e_2^{NC}).

Thus, the problem for non-cooperators is:

$$\max_{e_1} \int_0^{\infty} (f_1(e_1) - D_1(S))e^{-\rho t} dt, \tag{4.1}$$

and

$$\max_{e_2} \int_0^{T^N} (f_2(e_2) - d_2 - (S))e^{-\rho t} dt + \int_{T^N}^{\infty} (f_2(e_2) - D_2(S))e^{-\rho(t-T^N)} dt \tag{4.2}$$

subject to: $\dot{S}(t) = \mu\left(\sum_1^N e_i(t)\right) - \delta S(t), \quad S(0) = S_0,$

For cooperators, the joint maximization is:

$$\begin{aligned} \max_{e_1,e_2} \int_0^{T^C} & (n_1 f_1(e_1) + n_2 f_2(e_2) - n_1 D_1(S) - n_2 d_2(S) + Ext(n_1, n_2))e^{-\rho t} dt + \\ \int_{T^C}^{\infty} & (n_1 f_1(e_1) + n_2 f_2(e_2) - n_1 D_1(S) - n_2 D_2(S) + Ext(n_1, n_2))e^{-\rho(t-T^C)} dt, \end{aligned} \tag{4.3}$$

subject to: $\dot{S}(t) = \mu\left(\sum_1^N e_i(t)\right) - \delta S(t), \quad S(0) = S_0.$

In the maximization problems we call T^N and T^C the instants of time at which a developing country achieves the threshold to become developed, respectively in the non-cooperative and cooperative case.

4.2.1 Non-cooperative Emissions

To solve the problem, we use Dynamic Programming method. We proceed backward, solving first the problem on $[T^N, \infty)$. So, we have to solve first:

$$\max_{e_i^{NC}} \int_{T^N}^{\infty} \left(\alpha_i e_i^{NC} - \frac{1}{2}(e_i^{NC})^2 - \beta_i S \right) e^{-\rho(t-T^N)} dt,$$

subject to: $\dot{S}(t) = \mu \left(\sum_1^N e_i(t) \right) - \delta S(t), \quad S(0) = S_0.$

If we denote with $v(t, S)$ the value function of the problem, we can write the Hamilton-Jacobi-Bellman (HJB) equation

$$- v_t = \max_{e_i^{NC}} \left\{ \left(\alpha_i e_i - \frac{1}{2}(e_i^{NC})^2 - \beta_i S \right) e^{-\rho t} + v_S \left(\mu \sum_{j=1}^{N} e_j - \delta S \right) \right\} \tag{4.4}$$

Solving the first order conditions in (4.4), we obtain an expression for the non-cooperative emissions

$$e_i^{NC}(t) = \alpha_i + \mu v_S e^{\rho t}.$$

Let us assume that value function $v(t, S)$ is linear in S

$$v(t, S) = (KS + L)e^{-\rho t},$$

with partial derivatives $v_t = -\rho(KS + L)e^{-\rho t}$ and $v_S = Ke^{-\rho t}$. So, emissions for a player i outside coalition are given by

$$e_i^{NC}(t) = \alpha_i + K\mu.$$

In order to find an expression for the parameter K, we substitute v_t, v_S and e_i inside (4.4)

$$\rho(KS + L)e^{-\rho t} = \left[\alpha_i(\alpha_i + K\mu) - \frac{1}{2}(\alpha_i + K\mu)^2 - \beta_i S \right] e^{-\rho t} + K \left(\mu \sum_{j=1}^{N} e_j - \delta S \right) e^{-\rho t}$$

$$\rho(KS + L) = \alpha_i^2 + \alpha_i K\mu - \frac{1}{2}\alpha_i^2 - \frac{1}{2}K^2\mu^2 - \alpha_i K\mu - \beta_i S + K\mu \sum_{j=1}^{N} e_j - K\delta S$$

$$\rho KS + \rho L = -(\beta_i + K\delta)S + \frac{1}{2}\alpha_i^2 - \frac{1}{2}K^2\mu^2 + K\mu\sum_{j=1}^{N} e_j.$$

By the principle of identity of polynomials, we can write the equation

$$\rho K = -\beta_i - K\delta,$$

from which we have that $K = -\frac{\beta_i}{\rho+\delta}$.

Finally, we have the expression of non-cooperative emissions

$$e_i^{NC}(t) = \alpha_i - \mu\frac{\beta_i}{\rho+\delta}, \qquad i = 1, 2. \tag{4.5}$$

We proceed now to find the non-cooperative emissions for $t \in [0, T^N]$.

Since the functional forms for players of kind 1 is the same in the entire period, we have that the emissions of developed countries are the same in $[0, T^N]$:

$$e_1^{NC}(t) = \alpha_1 - \mu\frac{\beta_1}{\rho+\delta}, \qquad \forall t \geq 0.$$

For devoloping countries we have to consider the different damage-cost function, so the problem is:

$$\max_{e_2^{NC}} \int_0^{T^N} \left(\alpha_2 e_2^{NC} - \frac{1}{2}(e_2^{NC})^2 - \gamma\frac{t}{T^N}\beta_2 S\right)e^{-\rho t}dt,$$

subject to: $\dot{S}(t) = \mu\left(\sum_1^N e_i(t)\right) - \delta S(t), \quad S(0) = S_0.$

In this case the HJB equation is given by

$$-v_t = \max_{e_2}\left\{\left(\alpha_2 e_2 - \frac{1}{2}e_2^2 - \gamma\frac{t}{T^N}\beta_2 S\right)e^{-\rho t} + v_S\left(\mu\sum_{j=1}^{N} e_j - \delta S\right)\right\}. \tag{4.6}$$

As usual, we derive the first order conditions from maximization in (4.6)

$$(\alpha_2 - e_2)e^{-\rho t} + v_s\mu = 0,$$

then

$$e_2^{NC}(t) = \alpha_2 + v_S\mu e^{\rho t}.$$

For the value function $v(t,S)$ we assume

$$v(t,S) = [x(t)S + y(t)]e^{-\rho t},$$

whose partial derivatives are: $v_t = [(x'(t) - \rho x(t))S + y'(t) - \rho y(t)]e^{-\rho t}$ and $v_S = x(t)e^{-\rho t}$.
The emissions are given by

$$e_2^{NC}(t) = \alpha_2 + \mu x(t).$$

Now, we have to substitute v_t, v_S and e_2 inside (4.6)

$$-[(x'(t) - \rho x(t))S + y'(t) - \rho y(t)]e^{-\rho t} = \left[\alpha_2(\alpha_2 + \mu x(t)) - \frac{1}{2}(\alpha_2 + \mu x(t))^2 - \gamma \frac{t}{T^N}\beta_2 S\right]e^{-\rho t} + \\ + x(t)\left(\mu \sum_{j=1}^{N} e_j - \delta S\right)e^{-\rho t}.$$

Rearranging respect S and by the principle of identity of polynomials, we can write the differential equation

$$\begin{cases} -x'(t) + (\rho + \delta)x(t) = -\gamma \frac{t}{T^N}\beta_2, \\ x(T^N) = -\frac{\beta_2}{\rho+\delta}. \end{cases}$$

The solution $x(t)$ is given by

$$x(t) = -\frac{\beta_2}{T^N(\rho+\delta)^2}\left[\gamma\left(1 + t(\rho+\delta) - e^{(\rho+\delta)(t-T^N)}\right) + e^{(\rho+\delta)(t-T^N)}T^N(\rho+\delta)(1-\gamma)\right],$$

and leads us to the developing countries' non-cooperative emissions, for $t \in [0, T^N]$:

$$e_2^{NC}(t) = \alpha_2 - \mu \frac{\beta_2}{T^N(\rho+\delta)^2}\left[\gamma\left(1 + t(\rho+\delta) - e^{(\rho+\delta)(t-T^N)}\right) + T^N(1-\gamma)(\rho+\delta)e^{(\rho+\delta)(t-T^N)}\right] \tag{4.7}$$

4.2.2 Cooperative Emissions

Consider now, the $n_1 + n_2$ cooperators.
As for the non-cooperative case, we proceed backward. So, first of all solve the problem for $t \in [T^C, \infty)$:

$$\max_{e_1^C, e_2^C} \int_{T^C}^{\infty} \left[n_1\left(\alpha_1 e_1^C - \frac{1}{2}(e_1^C)^2\right) + n_2\left(\alpha_2 e_2^C - \frac{1}{2}(e_2^C)^2\right) - n_1\beta_1 S - n_2\beta_2 S + s_1 n_1 + s_2 n_2\right]e^{-\rho(t-T^C)}dt,$$

subject to: $\dot{S}(t) = \mu\Big(n_1 e_1^C(t) + n_2 e_2^C(t) + (N_1 - n_1)e_1^{NC}(t) + (N_2 - n_2)e_2^{NC}(t)\Big) - \delta S(t)$, $S(0) = S_0$.

We want to highlight that the externality have no effects on the determination of the emissions. Let $v(t,S)$ be the value function, the HJB equation is

$$-v_t = \max_{e_1^C, e_2^C}\Bigg\{\Bigg[n_1\Big(\alpha_1 e_1^C - \frac{1}{2}(e_1^C)^2\Big) + n_2\Big(\alpha_2 e_2^C - \frac{1}{2}(e_2^C)^2\Big) - n_1\beta_1 S - n_2\beta_2 S\Bigg]e^{-\rho t} + \\ + v_S\Bigg[\mu\Big(n_1 e_1^C + n_2 e_2^C + (N_1 - n_1)e_1^{NC} + (N_2 - n_2)e_2^{NC}\Big) - \delta S\Bigg] + (s_1 n_1 + s_2 n_2)e^{-\rho t}\Bigg\}. \tag{4.8}$$

The first order conditions in (4.8) are given by

$$\begin{cases} n_1[(\alpha_1 - e_1^C)e^{-\rho t} + \mu v_S] = 0, \\ n_2[(\alpha_1 - e_2^C)e^{-\rho t} + \mu v_S] = 0, \end{cases}$$

from which we can derive the expressions for emissions

$$e_1^C(t) = \alpha_1 + \mu v_S e^{\rho t}, \qquad e_2^C(t) = \alpha_2 + \mu v_S e^{\rho t}.$$

The steps are the same: we choose a guess for value function, then we substitute its partial derivatives and the emissions in equation (4.8). So, take

$$v(t,S) = (AS + B)e^{-\rho t},$$

whose partial derivatives are: $v_t = -\rho(AS+B)e^{-\rho t}$ and $v_S = Ae^{-\rho t}$.
Cooperative emissions, than, are given by

$$e_1^C(t) = \alpha_1 + \mu A, \qquad e_2^C(t) = \alpha_2 + \mu A.$$

Substituting in (4.8), we can derive an expression for parameter A

$$\rho A = -n_1\beta_1 - n_2\beta_2 - A\delta,$$

so that, $A = -\frac{n_1\beta_1 + n_2\beta_2}{\rho+\delta}$. Than, the emission solutions for cooperative countries, for $t \in [T^C, \infty)$, are given by

$$e_i^C(t) = \alpha_i - \mu\frac{n_1\beta_1 + n_2\beta_2}{\rho + \delta}, \qquad i = 1, 2. \tag{4.9}$$

Now, we can proceed to find the feedback Nash equilibrium for cooperative players in the interval $[0, T^C]$.

The different damage-cost function implies that the cooperative solutions solve:

$$\max_{e_1^C, e_2^C} \int_0^{T^C} \left[n_1 \left(\alpha_1 e_1^C - \frac{1}{2}(e_1^C)^2 \right) + n_2 \left(\alpha_2 e_2^C - \frac{1}{2}(e_2^C)^2 \right) - n_1 \beta_1 S - n_2 \gamma \frac{t}{T^C} \beta_2 S + s_1 n_1 + s_2 n_2 \right] e^{-\rho t} dt,$$

subject to: $\dot{S}(t) = \mu \Big(n_1 e_1^C(t) + n_2 e_2^C(t) + (N_1 - n_1) e_1^{NC}(t) + (N_2 - n_2) e_2^{NC}(t) \Big) - \delta S(t)$, $S(0) = S_0$. If we denote with $v(t, S)$ the value function, than the HJB equation is

$$-v_t = \max_{e_1^C, e_2^C} \Bigg\{ \left[n_1 \left(\alpha_1 e_1^C - \frac{1}{2}(e_1^C)^2 \right) + n_2 \left(\alpha_2 e_2^C - \frac{1}{2}(e_2^C)^2 \right) - n_1 \beta_1 S - n_2 \gamma \frac{t}{T^C} \beta_2 S \right] e^{-\rho t} +$$
$$+ v_S \left[\mu \Big(n_1 e_1^C + n_2 e_2^C + (N_1 - n_1) e_1^{NC} + (N_2 - n_2) e_2^{NC} \Big) - \delta S \right] + (s_1 n_1 + s_2 n_2) e^{-\rho t} \Bigg\}. \tag{4.10}$$

As usual, we compute the first order conditions in (4.10), to obtain a characterization for cooperative emissions. So

$$\begin{cases} n_1[(\alpha_1 - e_1^C) e^{-\rho t} + v_S \mu] = 0, \\ n_2[(\alpha_2 - e_2^C) e^{-\rho t} + v_S \mu] = 0, \end{cases}$$

from which $e_i^C(t) = \alpha_i - v_S \mu$, $i = 1, 2$.

We need to give a guess for value function, and we choose, as in the previous cases, a linear function of S

$$v(t, S) = [g(t) S + z(t)] e^{-\rho t},$$

whose partial derivatives respect t and S are

$$v_t = [(g'(t) - \rho g(t)) S + z'(t) - z(t)] e^{-\rho t}, \qquad v_S = g(t) e^{-\rho t}.$$

The expression of v_S give us the emission solutions

$$e_i^C = \alpha_i + \mu g(t), \qquad i = 1, 2.$$

To conclude the computation of the Nash equilibrium, we need to find an expression for the function $g(t)$. The way is to substitute v_t, v_S and e_i, $i = 1, 2$, inside (4.10). With some algebra, and because of the continuity of value function, we have to solve the dynamical system

$$\begin{cases} -g'(t) + (\rho+\delta)g(t) = n_1\beta_1 + n_2\gamma\frac{t}{T^C}\beta_2, \\ g(T^C) = -\frac{n_1\beta_1+n_2\beta_2}{\rho+\delta}. \end{cases}$$

The system has a unique solution $g(t)$, as follows

$$g(t) = -\frac{n_1\beta_1}{\rho+\delta} - \frac{n_2\beta_2}{T^C(\rho+\delta)^2}\Big[\gamma\big(1+t(\rho+\delta)-e^{(\rho+\delta)(t-T^C)}\big)+T^C(1-\gamma)(\rho+\delta)e^{(\rho+\delta)(t-T^C)}\Big].$$

Finally, the emissions for cooperative players, when $t \in [0, T^C]$, are given by:

$$e_i^C(t) = \alpha_i - \mu\frac{n_1\beta_1}{\rho+\delta} - \mu\frac{n_2\beta_2}{T^C(\rho+\delta)^2}\Big[\gamma\big(1+t(\rho+\delta)-e^{(\rho+\delta)(t-T^C)}\big)+T^C(1-\gamma)(\rho+\delta)e^{(\rho+\delta)(t-T^C)}\Big], \tag{4.11}$$

where $i = 1, 2$. Nevertheless, we will see that externalities will effect the payoffs of the players.

4.3 Stability

In order to approach the stability analysis, we use the concept of self-enforcing agreements. The idea is due to [dAs+83], and they use this concept to study the stability of a cartel. Several times it is used to discuss the stability of environmental agreements. We want highlight that these conditions are more stringent and there are different papers that try to propose different ways to face the problem (see [Fin03] and [EF06]). The basic idea is that a coalition is stable if no one inside has an incentive to defect and no one outside has an incentive to join in. So, called w the pay-off of a player, a coalition of k players is stable if are verified:

$$w_i^C(k) \geq w_i^{NC}(k-1), \qquad w_j^{NC}(k) \geq w_j^C(k+1),$$

$\forall i \in C$ and $\forall j \in NC$. First condition is called *internal* stability, while the second is called *external* stability.

In our case, having two different kind of players, we need to adapt the definition, asking that internal and external stability are verified both for developed and developing countries. So, we have to find the values n_1 and n_2 that solve this system of four inequalities:

$$\begin{cases} w_1^C(n_1, n_2) \geq w_1^{NC}(n_1 - 1, n_2) \\ \\ w_2^C(n_1, n_2) \geq w_2^{NC}(n_1, n_2 - 1) \\ \\ w_1^{NC}(n_1, n_2) \geq w_1^C(n_1 + 1, n_2) \\ \\ w_2^{NC}(n_1, n_2) \geq w_2^C(n_1, n_2 + 1) \end{cases} \tag{4.12}$$

Our interest is to find a stable coalition in the short run, so we focus our analysis in the period $[0, T]$. Unfortunately, we are not able to solve the system analytically, due to the complexity and the high non linearity of the functions. So, we analyze the problem from a numerical point of view.

First of all, we have to select values for parameters. Given the similarities, we use the same calibration of [Bah+09], according to the MERGE [1]. This model divides world in nine region: Canada, Australia and New Zealand (CANZ); China; Eastern Europe and former Soviet Union (EEFSU); India; Japan; Mexico and OPEC countries (MOPEC); USA; Western Europe (WEUR) and the rest of the world (ROW). To make our analysis, we use Western Europe and India as representatives of developed countries and developing countries. So we have

$$\alpha_1 = 993, \qquad \alpha_2 = 334, \qquad \beta_1 = 0.310, \qquad \beta_2 = 0.063, \qquad \delta = 0.0171,$$
$$\mu = 1, \qquad \rho = 0.051, \qquad S_0 = 390000.$$

We make two different studies, one with $\gamma = 0$ and one with $\gamma = 1$. Moreover, we suppose to have a world with $N_1 = 50$ developed countries and $N_2 = 50$ developing countries. We want to test if the stability conditions (4.12) are verified for different combinations of n_1 and n_2. So, we first calculate the values of T^N and T^C, taking every time $\bar{Y}_2$ as a percentage of the maximal growth possible. After that, we solve the differential equations for $S(t)$ in the different configurations required by stability conditions.
Having all the elements we need, we can proceed with the simulation of stability. We evaluate for 64 different combinations of variables n_1 and n_2, defining $n_1 = 1 + 7i$ and $n_2 = 1 + 7j$, with $i, j = 1, ..., 7$. Our evaluation is on the positivity of stability conditions (4.12), written as difference between welfares

$$\begin{cases} w_1^C(n_1, n_2) - w_1^{NC}(n_1 - 1, n_2) \geq 0, \\ w_2^C(n_1, n_2) - w_2^{NC}(n_1, n_2 - 1) \geq 0, \\ w_1^{NC}(n_1, n_2) - w_1^C(n_1 + 1, n_2) \geq 0, \\ w_2^{NC}(n_1, n_2) - w_2^C(n_1, n_2 + 1) \geq 0. \end{cases}$$

Example 4.1. *We consider first the case without externality.*
Stability analysis is presented in figure 4.1 and 4.2 when $\gamma = 0$ and in figure 4.3 and 4.4 when $\gamma = 1$. Figures can be interpreted as follows: on the x axis we have the combinations of n_1 and n_2 given by the 64 permutations of the indeces i, j. So, $x = 1$ is given by $i = 0, j = 0$; $x = 2$ is given by $i = 0, j = 1$ and so on.
On the y axis there are the values of stability conditions, for each combination of n_1 and n_2. Then, the stability conditions are verified if the points lie in first quadrant,

[1] Model for Evaluating the Regional and Global Effects of Green House Gases reductions (see [MMR95]).

while are not verified if lie in the fourth quadrant. In the case $\gamma = 0$ *the internal stability is verified only for small coalition, but for those size there's not external stability. Increasing the size of the coalition leads internal instability. Then, it's not possible have a stable coalition of any size. When* $\gamma = 1$*, instead, the internal stability condition is never verified.*

So, for that choice of parameters we can't have a stable coalition without externality.

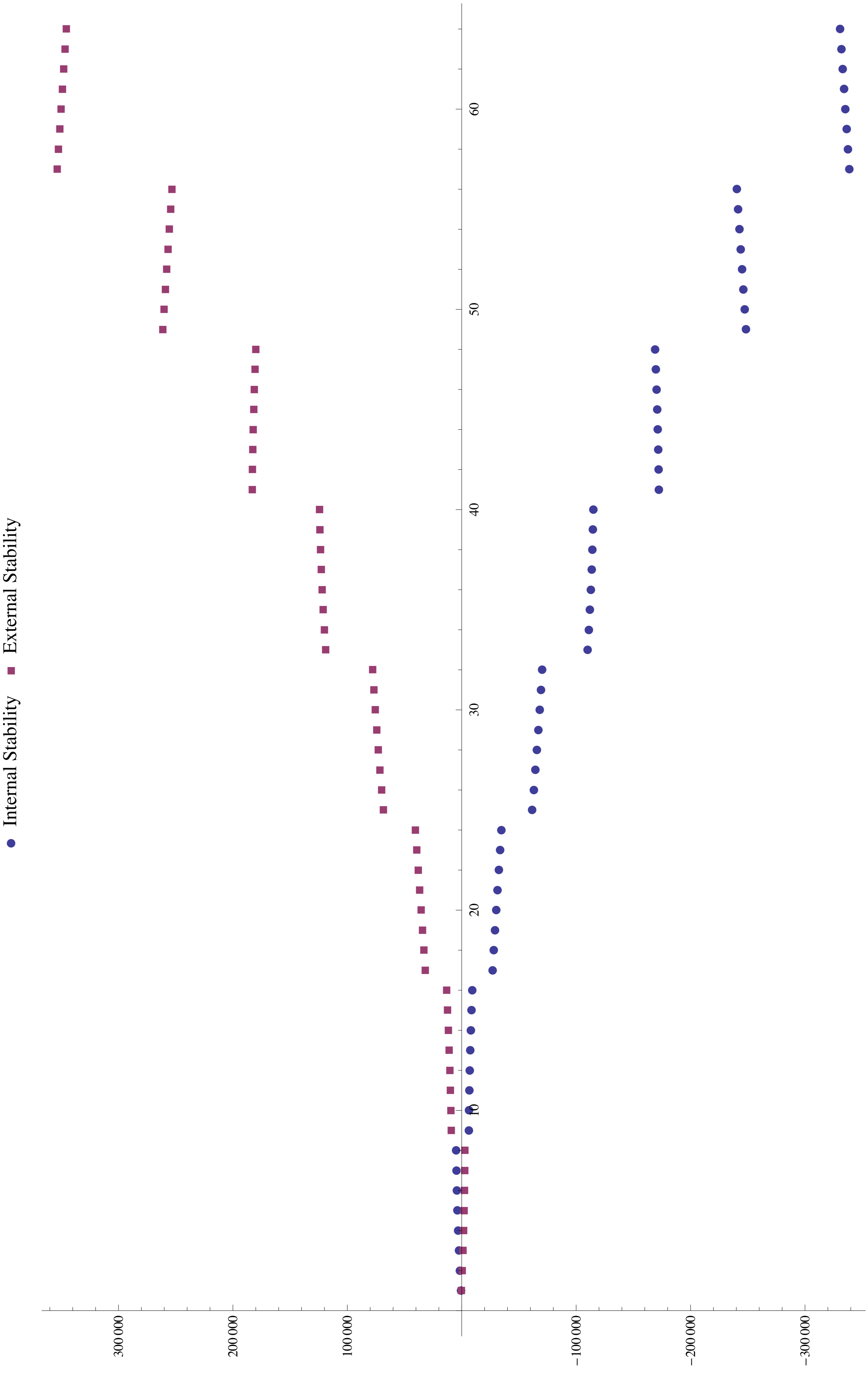

FIGURE 4.1: Example 4.1: Internal and External Stability for developed countries ($\gamma = 0$)

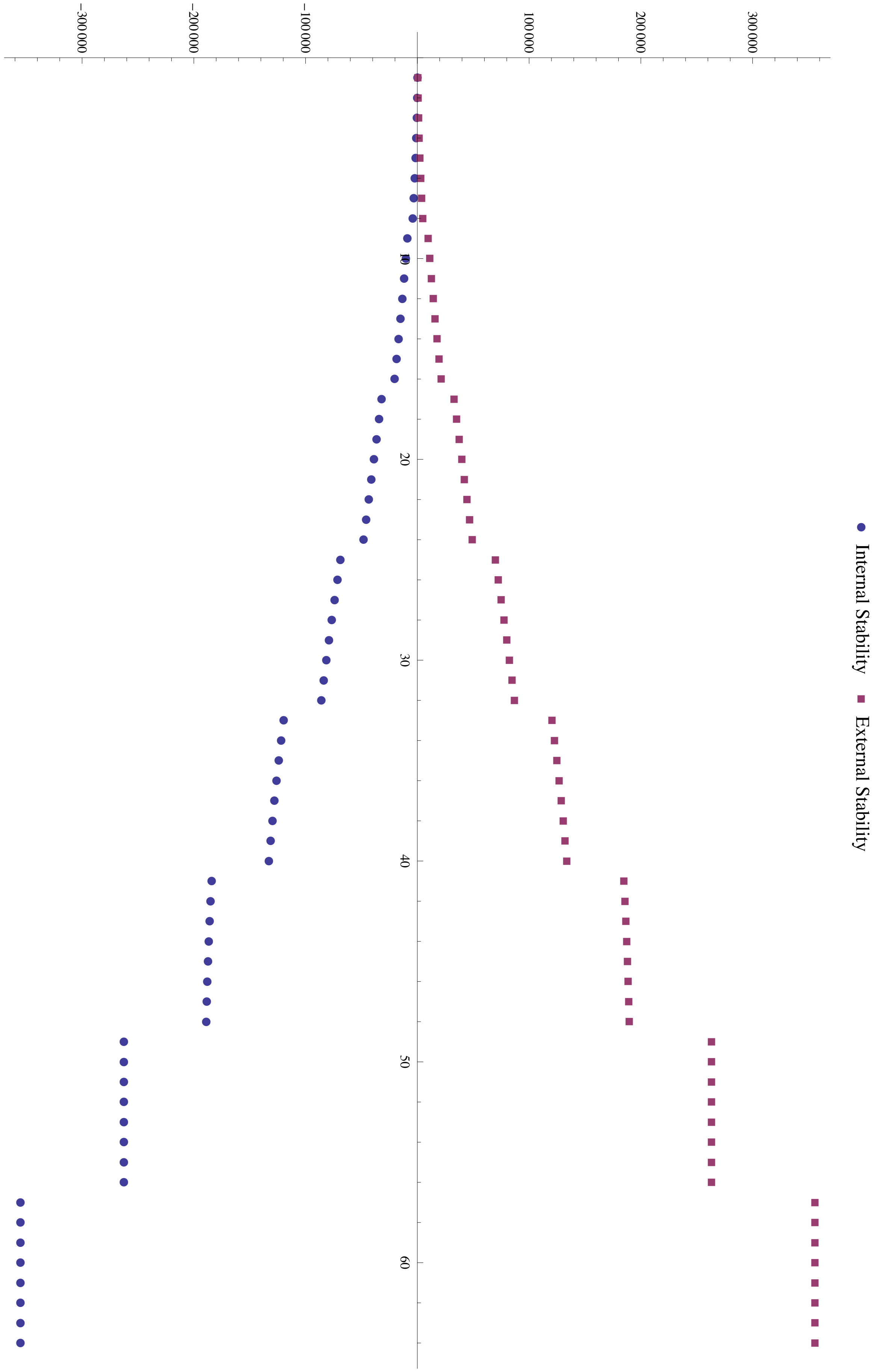

FIGURE 4.2: Example 4.1: Internal and External Stability for developing countries ($\gamma = 0$)

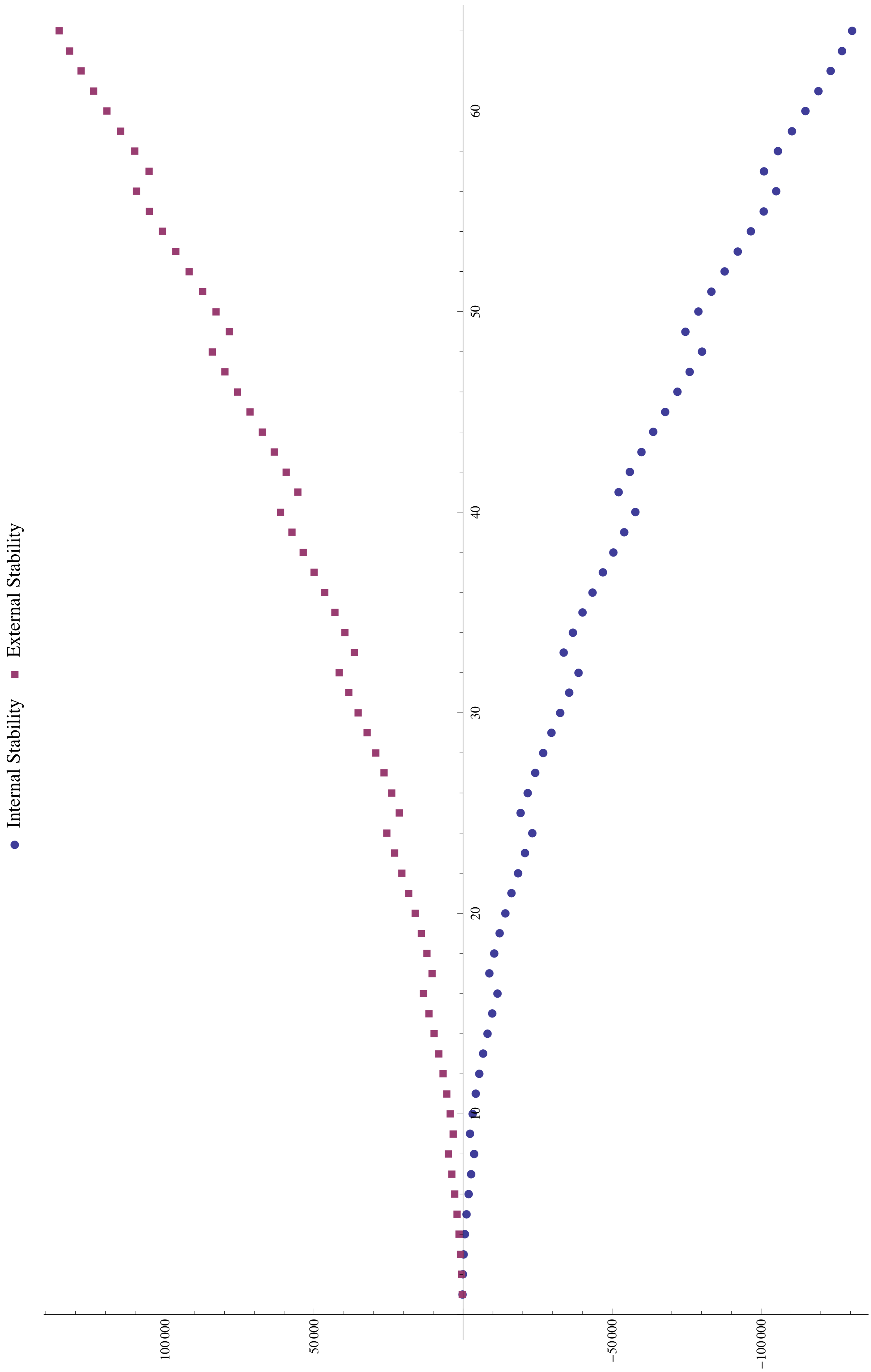

FIGURE 4.3: Example 4.1: Internal and External Stability for developed countries ($\gamma = 1$)

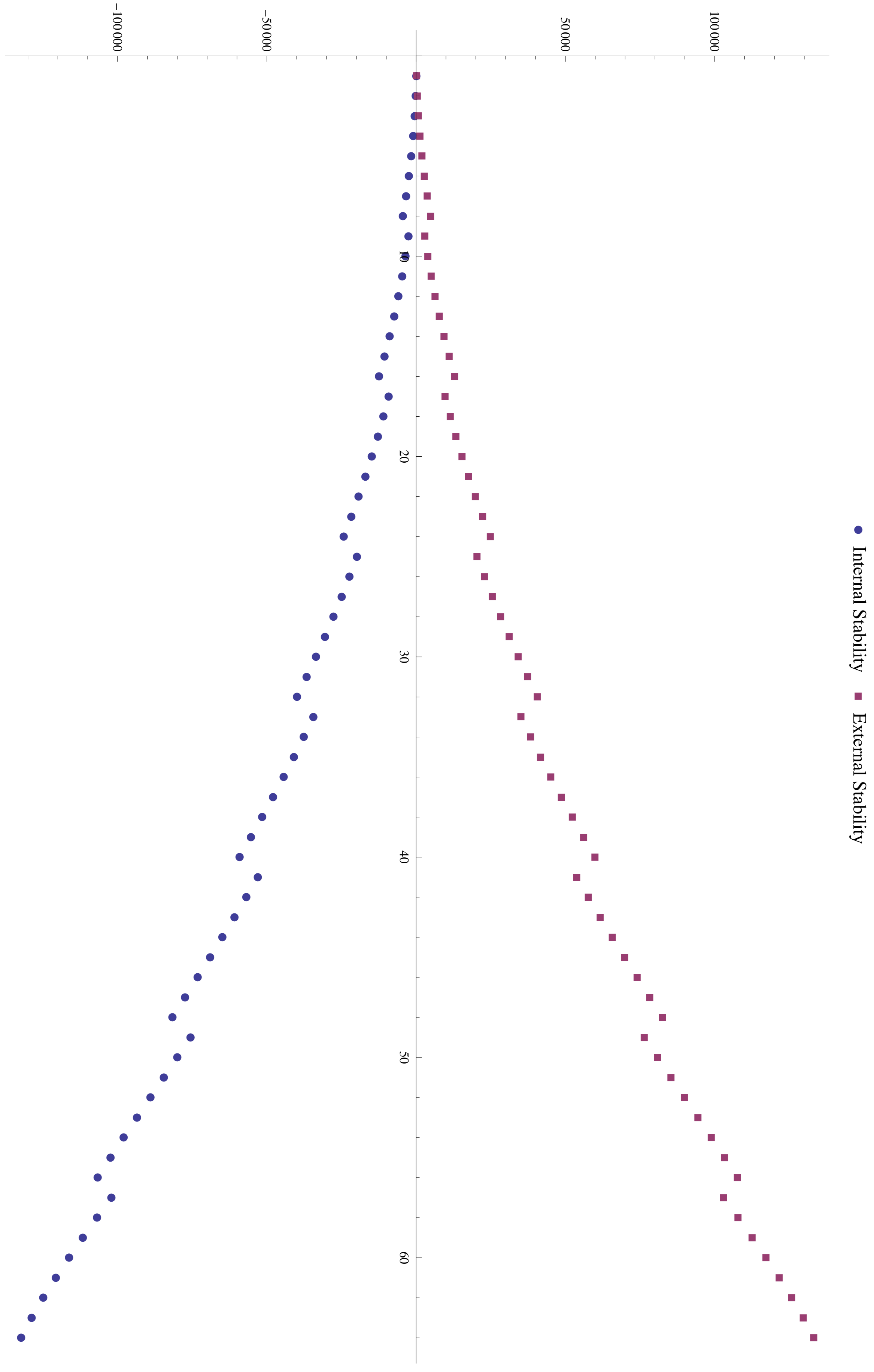

FIGURE 4.4: Example 4.1: Internal and External Stability for developing countries ($\gamma = 1$)

Example 4.2. *Now we come to the case of externality.*

What we observed here is that the solution are strictly related to the value of externality parameters, s_1 *and* s_2*. We found that if these parameters assume a value that is at least close to* 0.6 *of, respectively,* α_1 *and* α_2*, the only stable coalition is the grand coalition. Than, we assume*

$$s_1 = 595.8 \qquad\qquad s_2 = 200.4.$$

As we can see from figure 4.5 and 4.6, for $\gamma = 0$*, and from figure 4.7 and 4.8, for* $\gamma = 1$*, the internal stability is always verified, while we never have the external stability.*

So, all the players have the incentive to join in coalition.

On the other side, if the externality is not sufficiently large, than the positive effect is not enough to have a self-enforcing agreement.

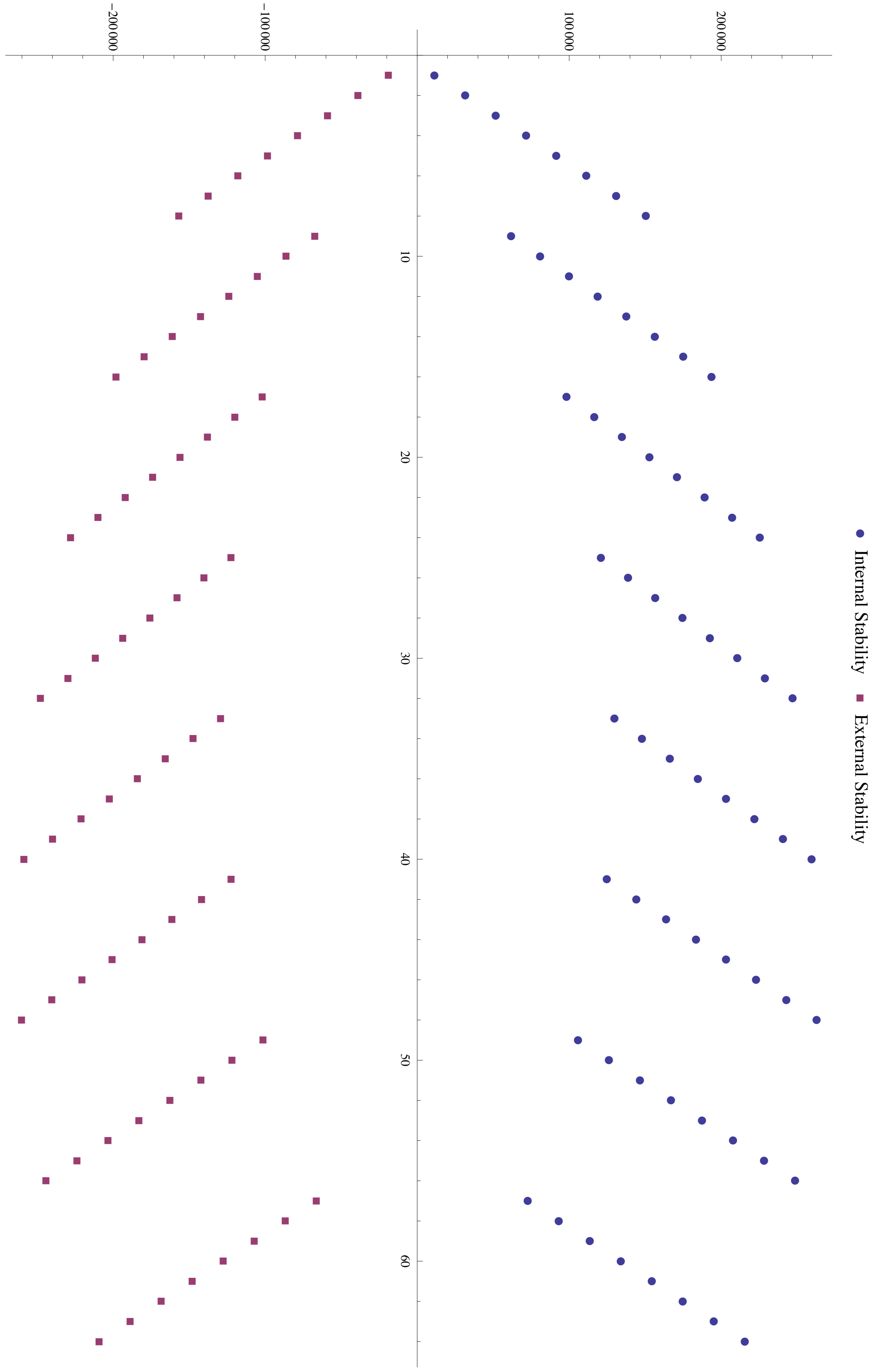

FIGURE 4.5: Example 4.2: Internal and External Stability for developed countries ($\gamma = 0$)

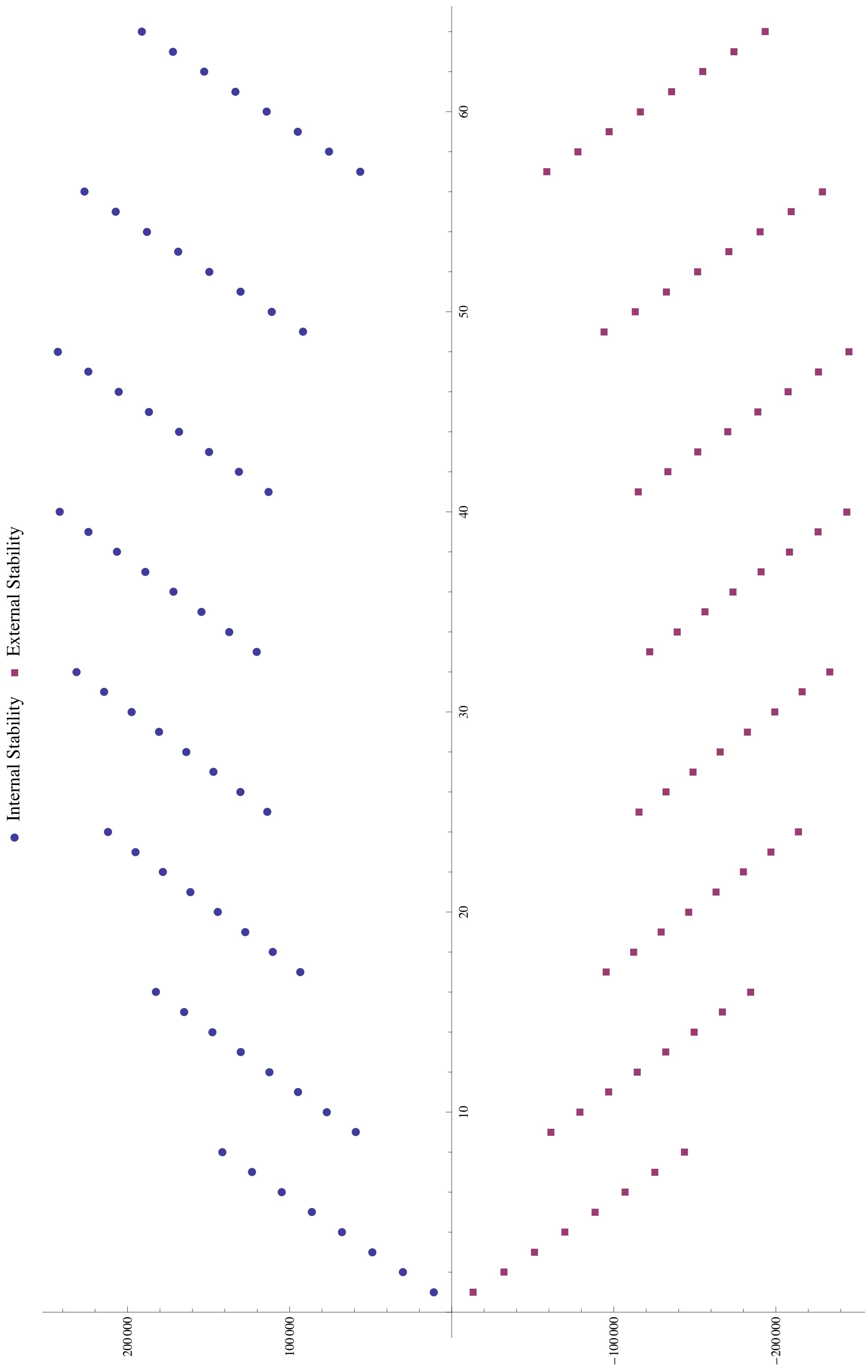

FIGURE 4.6: Example 4.2: Internal and External Stability for developing countries ($\gamma = 0$)

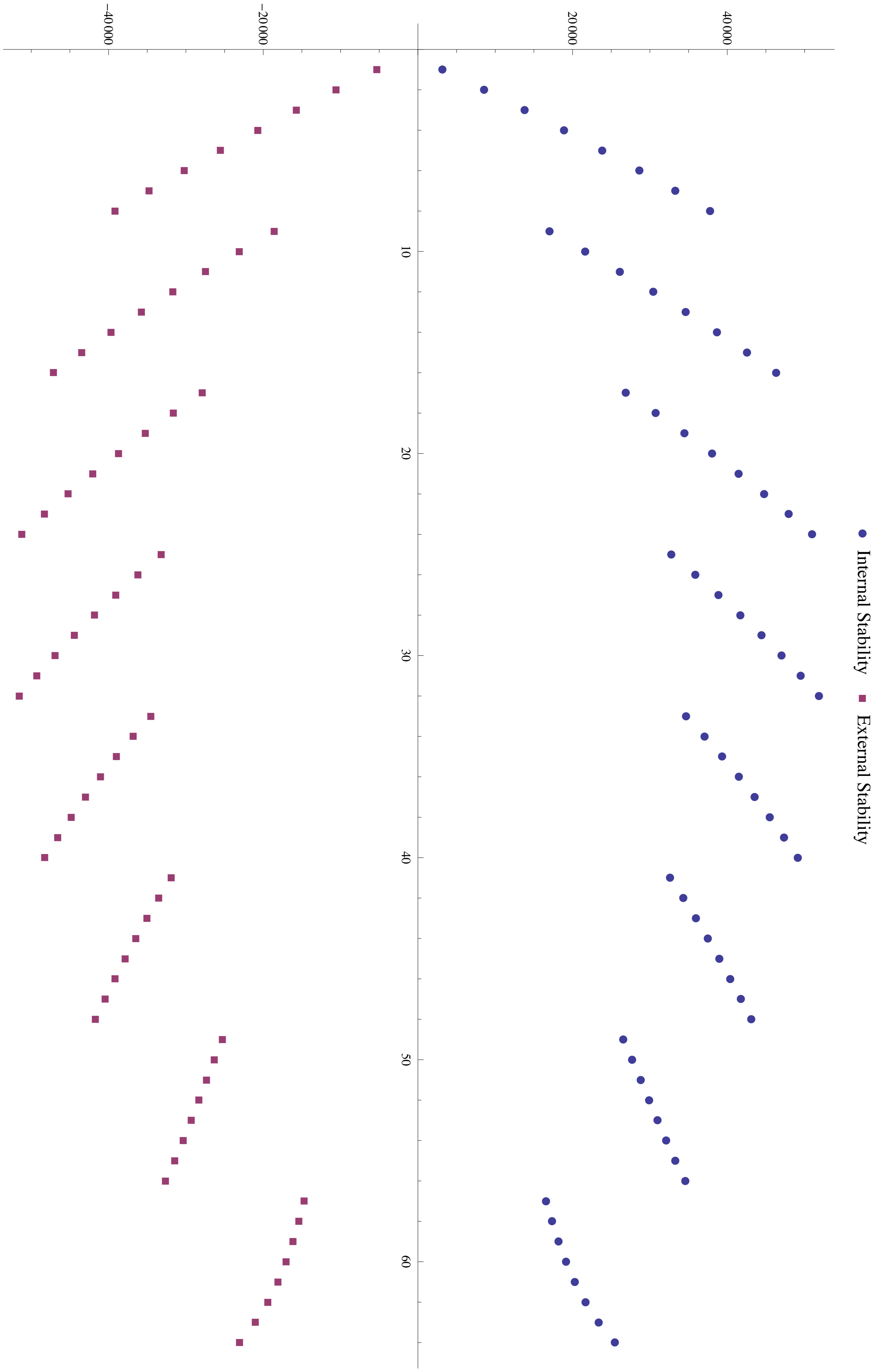

FIGURE 4.7: Example 4.2: Internal and External Stability for developed countries ($\gamma = 1$)

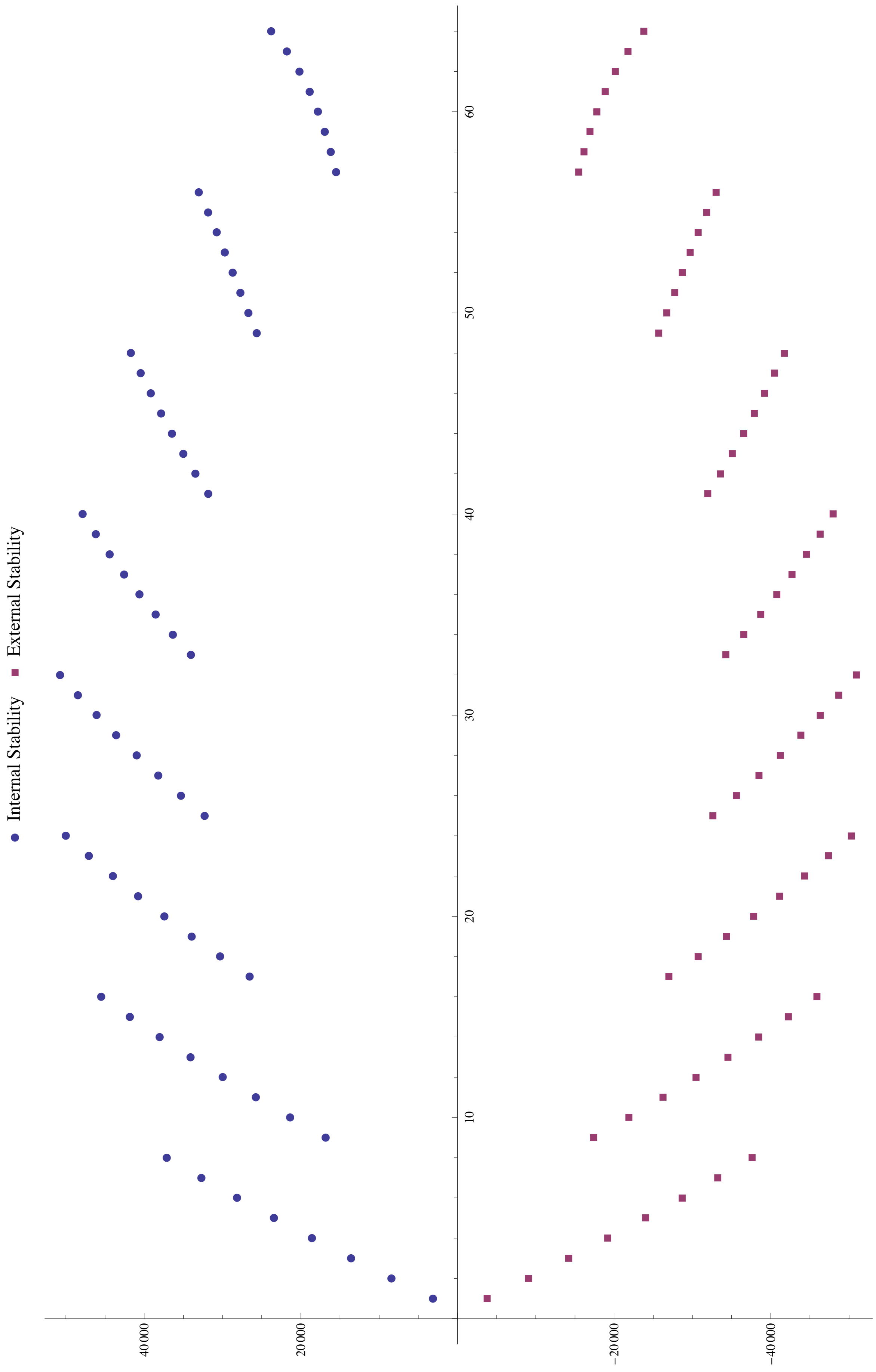

FIGURE 4.8: Example 4.2: Internal and External Stability for developing countries ($\gamma = 1$)

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